Chöông 6

GIAI GAN ÑUNG
PHÖÔNG TRÌNH VI PHAN

I. GIAI GAN ÑUNG PTVP CAP 1 :

Xet bai toan Cauchy : tìm nghiem y=y(x) cua phöông trình vi phan vôi gia trò ban ñau y_0

$$\begin{cases} y' = f(x, y), \ \ \forall x \in [a,b] \\ y(a) = y_0 \end{cases}$$

Cac phöông phap giai gan ñung :

- Cong thöc Euler
- Cong thöc Euler cai tien
- Cong thöc Runge-Kutta

1. Cong thöc Euler :

Ñe tìm nghiem gan ñung cua bai toan Cauchy ta chia ñoan [a,b] thanh n ñoan nho bang nhau vôi böôc h = (b-a)/n

$x_o = a,\ x_1 = x_0 + h,\ \ldots,\ x_k = x_0 + kh,\ \ldots,\ x_n = b$

Nghiem gan ñung cua bai toan la day $\{y_k\}$ gom cac gia trò gan ñung cua ham tai x_k

Ta co $y_k \approx y(x_k)$, $k = 0, n$

Gia sö bai toan co nghiem duy nhat y(x) co ñao ham ñen cap 2 lien tuc tren [a,b].

Khai trien Taylor ta co

$y(x_{k+1}) = y(x_k) + (x_{k+1}-x_k)\, y'(x_k) + (x_{k+1}-x_k)^2\, y''(\xi_k)/2$

vôi $\xi_k \in (x_k, x_{k+1})$

Cong thöc Euler :

$$y_{k+1} = y_k + h\, f(x_k, y_k)\ , k = 0, n-1$$

vôi $h = x_{k+1} - x_k$

Ví du : Dung cong thöc Euler tìm nghiem gan ñung cua bai toan Cauchy

$$\begin{cases} y' = y - x^2 + 1, \quad 0 \le x \le 1 \\ y(0) = 0.5 \end{cases}$$

vôi n = 5

Tính sai so biet nghiem chính xac la :

$$y(x) = (x+1)^2 - 0.5e^x$$

giai

ta co h = 0.2

$x_0 = 0$, $x_1 = 0.2$, $x_2 = 0.4$, $x_3 = 0.6$, $x_4 = 0.8$, $x_5 = 1$

Cong thöc Euler

$$\begin{cases} y_0 = 0.5 \\ y_{k+1} = y_k + h\,f(x_k, y_k) = y_k + 0.2\,(y_k - x_k^2 + 1) \end{cases}$$

k	x_k	y_k	$y(x_k)$	$\lvert y(x_k) - y_k \rvert$
0	0	0.5	0.5	0
1	0.2			
2	0.4			
3	0.6			
4	0.8			
5	1			

A = 0

B = 0.5

B = B + 0.2(B – A^2 + 1) : A=A+0.2:

(A+1)2-0.5eA:Ans-B

* <u>Nhan xet</u> : cong thöc Euler ñôn gian, nhöng sai so con lôn nen ít ñöôc sö dung

2. Cong thöc Euler cai tien :

$$\begin{cases} y_{k+1} = y_k + (k_1+k_2)/2 \\ k_1 = hf(x_k, y_k), \\ k_2 = hf(x_k+h, y_k + k_1) \end{cases} \qquad k = 0,1, ..., n-1$$

vôi $h = x_{k+1} - x_k$

Ví duï : Duøng coâng thöùc Euler caûi tieán tìm nghieäm gaàn ñuùng cuûa baøi toaùn Cauchy

$$\begin{cases} y' = y - x^2 + 1, & 0 \le x \le 1 \\ y(0) = 0.5 \end{cases}$$

vôùi n = 5

Tính sai soá bieát nghieäm chính xaùc laø :

$$y(x) = (x+1)^2 - 0.5e^x$$

giaûi

ta coù h = 0.2

$x_0 = 0$, $x_1 = 0.2$, $x_2 = 0.4$, $x_3 = 0.6$, $x_4 = 0.8$, $x_5 = 1$

Cong thöc Euler cai tien

$$\begin{cases} y_o = 0.5 \\ y_{k+1} = y_k + \ (k_1 + k_2)/2 \\ k_1 = 0.2(y_k - x_k^2 + 1) \\ k_2 = 0.2(y_k + k_1 - (x_k + 0.2)^2 + 1) \end{cases}$$

k	x_k	y_k	$y(x_k)$	$\lvert y(x_k) - y_k \rvert$
0	0	0.5	0.5	0
1	0.2			
2	0.4			
3	0.6			
4	0.8			
5	1			

$A = 0 \ \ (x_k)$

$B = 0.5 \ \ (y_k)$

$C = 0.2(B - A^2 + 1) :$

$D = 0.2(B + C - (A+0.2)^2 + 1):$

$B=B + (C+D)/2:$

$A=A+0.2:$

$(A+1)^2-0.5e^A$:Ans-B

3. Công thức Runge Kutta bậc 4 :

$$
\begin{cases}
y_{k+1} = y_k + \dfrac{1}{6}(K_1 + 2K_2 + 2K_3 + K_4) \\
K_1 = hf(x_k, y_k) \\
K_2 = hf\left(x_k + \dfrac{h}{2}, y_k + \dfrac{K_1}{2}\right) \\
K_3 = hf\left(x_k + \dfrac{h}{2}, y_k + \dfrac{K_2}{2}\right) \\
K_4 = hf(x_k + h, y_k + K_3)
\end{cases}
$$

Ví duï : Xeùt baøi toaùn Cauchy

$$\begin{cases} y' = 2.7xy + \cos(x+2.7y), & 1.2 \leq x \\ y(1.2) = 5.4 \end{cases}$$

Duøng coâng thöùc Runge-Kutta tính gaàn ñuùng y(1.5) vôùi böôùc h = 0.3

giaûi

Coâng thöùc Runge-Kutta baäc 4

$x_o = 1.2,\ y_o = 5.4$

$y_1 = y_0 + (K_1 + 2K_2 + 2K_3 + K_4)/6$

$K_1 = 0.3(2.7x_o y_o + \cos(x_o+2.7y_o))$

$K_2 = 0.3(2.7(x_o+0.3/2)(y_o+K_1/2) + \cos(x_o+0.3/2 + 2.7(y_o+K_1/2))$

$K_3 = 0.3(2.7(x_o+0.3/2)(y_o+K_2/2) + \cos(x_o+0.3/2 + 2.7(y_o+K_2/2))$

$K_4 = 0.3(2.7(x_o+0.3)(y_o+K_3) + \cos(x_o+0.3 + 2.7(y_o+K_3)$

Baím maùy ta ñöôïc

$K_1 = 4.949578057$ $K_2 = 8.367054617$

$K_3 = 10.33000627$ $K_4 = 19.41193853$

$y(1.5) = 15.69260639 \approx 15.6926$

Ví duï : Duøng coâng thöùc Runge-Kutta tìm nghieäm gaàn ñuùng cuûa baøi toaùn Cauchy

$$\begin{cases} y' = y - x^2 + 1, & 0 \leq x \leq 1 \\ y(0) = 0.5 \end{cases}$$

vôùi n = 5

Tính sai soá bieát nghieäm chính xaùc laø :

$$y(x) = (x+1)^2 - 0.5e^x$$

giaûi

ta coù h = 0.2

$x_0 = 0$, $x_1 = 0.2$, $x_2 = 0.4$, $x_3 = 0.6$, $x_4 = 0.8$, $x_5 = 1$

$A = 0 \ \ (x_k)$

$B = 0.5 \ \ (y_k)$

$C = 0.2(B - A^2 + 1)$:

$D = 0.2(B + C/2 - (A+0.1)^2 + 1)$:

$E = 0.2(B + D/2 - (A+0.1)^2 + 1)$:

$F = 0.2(B + E - (A+0.2)^2 + 1)$:

$B = B + (C+2D+2E+F)/6$:

$A = A+0.2$:

$(A+1)^2-0.5e^A$:Ans-B

Cong thöc Runge-Kutta bac 4

$$y_{k+1} = y_k + (K_1 + 2K_2 + 2K_3 + K_4)/6$$

$$K_1 = 0.2(y_k - x_k^2 + 1)$$

$$K_2 = 0.2\,[y_k + 0.1(y_k - x_k^2 + 1) - (x_k+0.1)^2 + 1\,]$$

$$= 0.2(1.1\,y_k - 1.1x_k^2 - 0.2x_k + 1.09)$$

$$K_3 = 0.2[\,y_k + 0.1(1.1y_k - 1.1x_k^2 - 0.2x_k + 1.09) - (x_k+0.1)^2 + 1\,]$$

$$= 0.2(1.11y_k - 1.11x_k^2 - 0.22x_k + 1.099)$$

$$K_4 = 0.2[\,y_k + 0.2(1.11y_k - 1.11x_k^2 - 0.22x_k + 1.099) - (x_k+0.2)^2 + 1\,]$$

$$= 0.2(1.222y_k - 1.222x_k^2 - 0.444x_k + 1.1798)$$

$$\begin{cases} y_0 = 0.5 \\ y_{k+1} = y_k + 0.2(6.642y_k - 6.642x_k^2 - 1.284x_k + 6.5578)/6 \end{cases}$$

k	x_k	y_k	$y(x_k)$	$\lvert y(x_k) - y_k \rvert$
0	0	0.5	0.5	0
1	0.2			
2	0.4			
3	0.6			
4	0.8			
5	1			

II. GIAI GAN ÑUNG HE PTVP :

Xet he phöông trình vi phan cap 1

$$\begin{cases} y'_1 = f_1(x, y_1, y_2, ..., y_m) \\ y'_2 = f_2(x, y_1, y_2, ..., y_m) \\ \quad\quad . . . \\ y'_m = f_m(x, y_1, y_2, ..., y_m) \end{cases}$$

vôi $a \leq x \leq b$ va thoa ñieu kien ban ñau

$$y_1(a) = \alpha_1,\ y_2(a) = \alpha_2,\ ....\ ,\ y_m(a) = \alpha_m$$

Nghiem $y = (y_1, y_2, \ldots, y_m)$

Ñe tìm nghiem gan ñung, ta chia ñoan [a,b] thanh n ñoan nho bang nhau vôi böôc h = (b-a)/n va cac ñiem chia

$x_o = a,\ x_1 = x_0 + h,\ \ldots,\ x_k = x_0 + kh,\ \ldots,\ x_n = b$

Nghiem gan ñung la day $\{ y_k = (y_{1\,k}, y_{2\,k}, \ldots, y_{m\,k}) \}$
vôi $y_{i\,k} \approx y_i(x_k)$

Cong thöc Euler :

$$y_{i\,k+1} = y_{i\,k} + h\, f_i(x_k, y_{1\,k}, \ldots, y_{m\,k})$$

$$\forall i=1..m;\ k = 0..\ n-1$$

Cong thöc Euler cai tien :

$$y_{i\,k+1} = y_{i\,k} + (K_{1\,i} + K_{2\,i}) / 2$$

$$K_{1\,i} = h\, f_i(x_k, y_{1\,k}, \ldots, y_{m\,k})$$

$$K_{2\,i} = h\, f_i(x_k+h, y_{1\,k}+K_{1\,1}, \ldots, y_{m\,k}+K_{1\,m})$$

$$\forall i=1,m;\ k = 0, n-1$$

Cong thöc Runge-Kutta bac 4 :

$$y_{i\,k+1} = y_{i\,k} + (K_{1\,i}+2K_{2\,i}+2K_{3\,i}+K_{4\,i}) / 6$$

$$K_{1\,i} = h\, f_i(x_k, y_{1\,k}, \ldots, y_{m\,k})$$

$$K_{2\,i} = h\, f_i(x_k+h/2, y_{1\,k}+K_{11}/2, \ldots, y_{m\,k}+K_{1\,m}/2)$$

$$K_{3\,i} = h\, f_i(x_k+h/2, y_{1\,k}+K_{21}/2, \ldots, y_{m\,k}+K_{2\,m}/2)$$

$$K_{4\,i} = h\, f_i(x_k+h, y_{1\,k}+K_{31}, \ldots, y_{m\,k}+K_{3\,m})$$

$$\forall i=1,m;\ k = 0, n-1$$

Ví dui : Söi duing coing thöic Euler giai gain ñuing hei pt vi phain

$$\begin{cases} y'_1 = 3y_1 + 2y_2 - (2x^2 + 1)e^{2x} \\ y'_2 = 4y_1 + y_2 + (x^2 + 2x - 4)\, e^{2x} \end{cases}$$

vôii 0 $\leq x \leq 0.5$

ñieiu kiein ban ñaiu $y_1(0)=y_2(0)=1$

böôic $h = 0.1$

So sainh vôii nghieim chính xaic

$$y_1(x) = 1/3e^{5x} - 1/3e^{-x} + e^{2x}$$

$$y_2(x) = 1/3e^{5x} + 2/3e^{-x} + x^2e^{2x}$$

Công thức Euler

$$\begin{cases} y_{1\,0} = 1 \\ y_{1\,k+1} = y_{1\,k} + h\,(3y_{1k} + 2y_{2\,k} - (2x_k^{\,2} + 1)e^{2x_k}) \end{cases}$$

$$\begin{cases} y_{2\,0} = 1 \\ y_{2\,k+1} = y_{2\,k} + h\,(4y_{1k} + y_{2\,k} + (x_k^{\,2} + 2x_k - 4)\,e^{2x_k}) \end{cases}$$

x_k	y_{1k}	$y_1(x_k)$	y_{2k}	$y_2(x_k)$
0	1	1	1	1
0.1				
0.2				
0.3				
0.4				
0.5				

$A=0$ (x)

$B=1$ (y_{1k})

$C=1$ (y_{2k})

$D=B + 0.1\ (3B + 2C - (2A^2 +1)e^{2A})$:

$C=C + 0.1\ (4B + C + (A^2 +2A -4)\ e^{2A})$:

$B=D$:

$A=A+0.1$

$A=0$

$e^{5A}/3-e^{-A}/3+e^{2A}$:

$e^{5A}/3+2/3e^{-A}/3+A^2e^{2A}$:

$A=A+0.1$

III. GIAÛI GAÀN ÑUÙNG PTVP CAÁP CAO:

Xeùt phöông trình vi phaân baäc m

$$y^{(m)}(x) = f(x, y, y', \ldots, y^{(m-1)}), \quad a \le x \le b$$

vôùi ñieàu kieän ban ñaàu

$$y(a) = \alpha_1,\ y'(a) = \alpha_2,\ \ldots,\ y^{(m-1)}(a) = \alpha_m$$

Ñaët $y_1 = y$, $y_2 = y'$, $y_3 = y''$, ... , $y_m = y^{(m-1)}$

Ta chuyeån phöông trình vi phaân baäc m veà heä m phöông trình vi phaân caáp 1

$$\begin{cases} y'_1 = y_2 \\ y'_2 = y_3 \\ \ldots \\ y'_{m-1} = y_m \\ y'_m = f(x, y, y', \ldots, y^{(m-1)}) \end{cases}$$

vôùi ñieàu kieän ban ñaàu

$$y_1(a) = \alpha_1,\ y_2(a) = \alpha_2,\ \ldots,\ y_m(a) = \alpha_m,$$

Ví duï : Söû duïng coâng thöùc Euler giaûi gaàn ñuùng pt vi phaân caáp 2

$$y'' - 2y' + 2y = \sin x\, e^{2x},\ 0 \le x \le 0.5$$

ñieàu kieän ban ñaàu

$$y(0) = -0.4,\ y'(0) = -0.6$$

vôùi böôùc h = 0.1

So saùnh vôùi nghieäm chính xaùc bieát nghieäm CX

$$y_1(x) = 0.2e^{2x}(\sin x - 2\cos x)$$

$$y_2(x) = 0.2e^{2x}(4\sin x - 3\cos x) = y'$$

ñat $y_1 = y$, $y_2 = y'$ chuyen pt ve he

$$\begin{cases} y'_1 = y_2 \\ y'_2 = \sin x\, e^{2x} - 2y_1 + 2y_2 \end{cases}$$

ñieu kien $y_1(0) = -0.4$, $y_2(0) = -0.6$

Cong thöc Euler

$$\begin{cases} y_{1\,0} = -0.4 \\ y_{1\,k+1} = y_{1\,k} + 0.1\, y_{2k} \end{cases}$$

$$\begin{cases} y_{2\,0} = -0.6 \\ y_{2\,k+1} = y_{2\,k} + 0.1\,(\sin x_k e^{2x_k} - 2y_{1k} + 2y_{2\,k}) \end{cases}$$

x_k	$y_{1\,k}$	$y_1(x_k)$	$y_{2\,k}$	$y_2(x_k)=y'(x_k)$
0	-0.4	-0.4	-0.6	-0.6
0.1				
0.2				
0.3				
0.4				
0.5				

A=0

B=-0.4

C=-0.6

D=B+0.1C

$C=C+0.1(\sin A e^{2A} - 2B + 2C)$

B=D

A=A+0.1