Chöông 5

TÍNH GAN ÑUNG

ÑAO HAM VA TÍCH PHAN

I. TÍNH GAN ÑUNG ÑAO HAM :

Cho ham $y = f(x)$ va bang so

x	x_o	x_1	x_2	. . .	x_n
y	y_o	y_1	y_2	. . .	y_n

Ñe tính gan ñung ñao ham, ta xap xæ ham bang ña thöc noi suy Lagrange $L_n(x)$

Ta co

$$f'(x) \approx L'_n(x)$$

$$f''(x) \approx L''_n(x)$$

1. TH bang chæ co 2 ñiem nut :

x	x_0	x_1
y	y_0	y_1

$h = x_1 - x_0$

$y_0 = f(x_0)$

$y_1 = f(x_1) = f(x_0+h)$

Ña thöc noi suy Lagrange

$$L_n(x) = \frac{(x - x_1)}{(x_0 - x_1)} y_0 + \frac{(x - x_0)}{(x_1 - x_0)} y_1$$

$$= \frac{(x - x_0)}{h} y_1 - \frac{(x - x_1)}{h} y_0$$

Do ño vôi moi $x \in [x_0, x_1]$ ta co

$$f'(x) \approx \frac{y_1 - y_0}{h} = \frac{f(x_0 + h) - f(x_0)}{h}$$

❖ Cong thöc sai phan tien :

$$f'(x_0) \approx \frac{f(x_0 + h) - f(x_0)}{h}$$

❖ Cong thöc sai phan lui :

$$f'(x_1) \approx \frac{y_1 - y_0}{h}$$

Thay x_1 bang x_0

$$f'(x_0) \approx \frac{f(x_0) - f(x_0 - h)}{h}$$

❖ Cong thöc sai so :

$$\Delta = \frac{M_2 h}{2} \quad \text{vôi} \quad M_2 = \max_{x\in[x_0,x_1]} |f''(x)|$$

❖ Ví du : Cho ham f(x) = ln x. Tính Xap xæ f'(1.8) va sai so vôi h = 0.1, 0.01, 0.001

giai

Ta co
$$f'(1.8) \approx \frac{f(1.8+h) - f(1.8)}{h}$$

$$f''(x) = -\frac{1}{x^2} \quad \Rightarrow M_2 = \max|f''(x)| = \frac{1}{1.8^2}$$

Sai so
$$\Delta = \frac{h}{2(1.8)^2}$$

h	f’(1.8)	Δ
0.1	0.540672212	0.016
0.01	0.554018037	0.16×10^{-2}
0.001	0.555401292	0.16×10^{-3}

2. TH bang co 3 ñiem nut cach ñeu :

x	x_0	x_1	x_2
y	y_0	y_1	y_2

$h = x_2 - x_1 = x_1 - x_0$

$y_0 = f(x_0)$

$y_1 = f(x_1) = f(x_0+h)$

$y_2 = f(x_2) = f(x_0+2h)$

Ña thöc noi suy Lagrange

$$L_n(x) = \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)}y_0 + \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)}y_1 + \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)}y_2$$

$$= \frac{(x-x_0)(x-x_1)}{2h^2}y_2 - \frac{(x-x_0)(x-x_2)}{h^2}y_1 + \frac{(x-x_1)(x-x_2)}{2h^2}y_0$$

Do ño vôi moi $x \in [x_0, x_2]$ ta co

$$f\,'(x) \approx \frac{(x-x_0)}{2h^2}(y_2 - 2y_1) + \frac{(x-x_1)}{2h^2}(y_2 + y_0) + \frac{(x-x_2)}{2h^2}(y_0 - 2y_1)$$

$$f\,''(x) \approx \frac{(y_2 - 2y_1 + y_0)}{h^2}$$

Suy ra ñaio ham cap 1

$$f'(x_0) \approx \frac{(-3y_0 + 4y_1 - y_2)}{2h}$$

$$f'(x_1) \approx \frac{(y_2 - y_0)}{2h}$$

$$f'(x_2) \approx \frac{(y_0 - 4y_1 + 3y_2)}{2h}$$

Cong thöc thöi 1 goi la <u>cong thöc sai phan tien</u>

$$\boxed{f'(x_0) \approx \frac{-3f(x_0) + 4f(x_0 + h) - f(x_0 + 2h)}{2h}}$$

Cong thöc thö 2 goi la cong thöc sai phan höông tam thöông viet döôi dang (thay $x_1 = x_0$)

$$f'(x_0) \approx \frac{f(x_0 + h) - f(x_0 - h)}{2h}$$

Cong thöc thö 3 goi la cong thöc sai phan lui thöông viet döôi dang (thay $x_2 = x_0$)

$$f'(x_0) \approx \frac{f(x_0 - 2h) - 4f(x_0 - h) + 3f(x_0)}{2h}$$

❖ Cong thöc sai so :

$$\Delta = \frac{M_3 h^2}{6} \quad \text{vôi} \quad M_3 = \max_{x \in [x_0, x_2]} |f'''(x)|$$

ñao ham cap 2

$$f''(x_1) \approx \frac{(y_2 - 2y_1 + y_0)}{h^2}$$

Thay $x_1 = x_0$ ta ñöôc

$$f''(x_0) \approx \frac{f(x_0 + h) - 2f(x_0) + f(x_0 - h)}{h^2}$$

- Cong thöc sai so :

$$\Delta = \frac{M_4 h^2}{12} \quad \text{vôi} \quad M_4 = \max_{x \in [x_0, x_2]} |f^{(4)}(x)|$$

❖ Ví duï : Cho haøm $f(x) = \ln x - 2/x^3$.

a. Duøng coâng thöùc sai phaân höôùng taâm, tính xaáp xæ f'(3) vôùi h = 0.1, 0.01, 0.001

b. Tính xaáp xæ f"(3) vôùi h = 0.1, 0.01, 0.001

giaûi

$$f'(3) \approx \frac{f(3+h) - f(3-h)}{2h}$$

h	f'(3)
0.1	
0.01	
0.001	

$$f''(3) \approx \frac{f(3+h) - 2f(3) + f(3-h)}{h^2}$$

h	f''(3)
0.1	
0.01	
0.001	

II. TÍNH GAN ÑUNG TÍCH PHAN :

Cho ham f(x) xac ñònh va kha tích tren [a,b].
Ta can tính gan ñung tích phan :

$$I = \int_a^b f(x)dx$$

Ta phan hoach ñoan [a,b] thanh n ñoan bang nhau vôi böôc h = (b-a)/n

$x_o = a,\ x_1 = x_0 + h,\ \ldots,\ x_k = x_0 + kh,\ \ldots,\ x_n = b$

Xap xæ f(x) bang ña thöc noi suy Lagrange

Ña thöc Lagrange trong TH cac ñiem cach ñeu

$$L_n(x) = q(q-1)\ldots(q-n)\sum_{k=0}^{n} \frac{(-1)^{n-k}}{k!(n-k)!(q-k)} y_k$$

$$\text{vôi } q = \frac{x-a}{h}$$

$$I \approx \int_a^b L_n(x)dx = \sum_{k=0}^{n} \int_a^b \frac{(-1)^{n-k} q(q-1)\ldots(q-n)}{k!(n-k)!(q-k)} y_k \, dx$$

$$= \sum_{k=0}^{n} \int_0^n \frac{(-1)^{n-k} q(q-1)\ldots(q-n)}{k!(n-k)!(q-k)} \frac{(b-a)}{n} dq \; y_k$$

$$I \approx I^* = (b-a)\sum_{k=0}^{n} H_k y_k$$

$$\text{vôi} \quad H_k = \frac{(-1)^{n-k}}{n\,k!(n-k)!}\int_0^n \frac{q(q-1)\ldots(q-n)}{(q-k)}\,dq$$

Cong thöc tren goi la cong thöc Newton-cotes, cac he so H_k goi la cac he so cotes.

He so cotes co cac tính chat sau :

$$\sum_{k=0}^{n} H_k = 1$$

$$H_{n-k} = H_k \quad k = \overline{0,n}$$

❖ Coâng thöùc sai soá :

$$\Delta = |I - I^*| \le \begin{cases} \dfrac{M_{n+1}h^{n+2}}{(n+1)!}\displaystyle\int_0^n |q(q-1)\ldots(q-n)|\,dq & \text{vôùi n leû} \\ \dfrac{M_{n+2}h^{n+3}}{(n+2)!}\displaystyle\int_0^n |q^2(q-1)\ldots(q-n)|\,dq & \text{vôùi n chaün} \end{cases}$$

$$M_{n+1} = \max_{x\in[a,b]} |f^{(n+1)}(x)| \text{ vaø } M_{n+2} = \max_{x\in[a,b]} |f^{(n+2)}(x)|$$

1. Cong thöc hình thang :

Xet n = 1, ta co h= b-a

$$I \approx (b\text{-}a)(H_o y_o + H_1 y_1)$$

$$H_0 = -\int_0^1 (q-1)dq = \frac{1}{2} \qquad \Rightarrow H_1 = H_0 = \frac{1}{2}$$

Vay $$I \approx \frac{(b-a)}{2}(y_0 + y_1) = \frac{(b-a)}{2}(f(a) + f(b))$$

❖ Cong thöc sai so :

$$\Delta \le \frac{M_2 h^3}{2!}\int_0^1 |q(q-1)|\,dq = \frac{M_2 h^3}{12}$$

❖ Cong thöc hình thang mô rong :

Ta phan hoach ñoan [a,b] thanh n ñoan bang nhau $[x_0, x_1], [x_1, x_2], \ldots, [x_{n-1}, x_n]$.

Ta co

$$I = \int_{x_0}^{x_1} f(x)dx + \int_{x_1}^{x_2} f(x)dx + \ldots + \int_{x_{n-1}}^{x_n} f(x)dx$$

$$= \frac{(x_1 - x_0)}{2}(y_0 + y_1) + \frac{(x_2 - x_1)}{2}(y_1 + y_2) + \ldots + \frac{(x_n - x_{n-1})}{2}(y_{n-1} + y_n)$$

$$= \frac{h}{2}(y_0 + y_1) + \frac{h}{2}(y_1 + y_2) + \ldots + \frac{h}{2}(y_{n-1} + y_n)$$

Vậy

$$I \approx \frac{h}{2}(y_0 + 2y_1 + \ldots + 2y_{n-1} + y_n)$$

❖ Công thức sai số :

$$\Delta \le n\frac{M_2 h^3}{12} = (b-a)\frac{M_2 h^2}{12}$$

2. Coäng thöùc Simpson :

Xeùt n = 2, ta coù h = (b-a)/2

$$I \approx (b-a)(H_o y_o + H_1 y_1 + H_2 y_2)$$

$$H_0 = \frac{1}{4}\int_0^1 (q-1)(q-2)dq = \frac{1}{6}$$

$$H_2 = H_0 = \frac{1}{6}$$

$$H_0 + H_1 + H_2 = 1 \quad \Rightarrow \quad H_1 = \frac{2}{3}$$

Vaäy $\quad I \approx \frac{(b-a)}{6}(y_0 + 4y_1 + y_2)$

❖ Coâng thöùc sai soá :

$$\Delta \le \frac{M_4 h^4}{4!}\int_0^1 |q^2(q-1)(q-2)|\,dq = \frac{M_4 h^5}{90}$$

❖ Coâng thöùc Simpson môû roäng :

Ñieàu kieän n phaûi chaün

Ta phaân hoaïch ñoaïn [a,b] thaønh n ñoaïn baèng nhau $[x_0, x_1], [x_1, x_2], \ldots, [x_{n-1}, x_n]$.