Chöông 4

NOÄI SUY VAØ XAÁP XÆ HAØM

I. ÑAT BAI TOAN :

Ñe tính gia trò cua mot ham lien tuc bat ky, ta co the xap xæ ham bang mot ña thöc, tính gia trò cua ña thöc tö ño tính ñöôc gia trò gan ñung cua ham

Xet ham y = f(x) cho dööôi dang bang so

x	x_o	x_1	x_2	. . .	x_n
y	y_o	y_1	y_2	. . .	y_n

- Cac gia trò x_k, k = 0, 1, .., n ñöôïc sap theo thöù töï tang dan goi la cac ñiem nut noi suy
- Cac gia trò $y_k = f(x_k)$ la cac gia trò cho tröôc cua ham tai x_k

Bai toan : xay döïng 1 ña thöùc $p_n(x)$ bac $\leq$n thoa ñieu kien $p_n(x_k) = y_k$, k=0,1,.. n. Ña thöùc nay goi la ña thöùc noi suy cua ham f(x).

II. ÑA THÖC NOY SUY LAGRANGE:

Cho ham $y = f(x)$ va bang so

x	x_o	x_1	x_2	. . .	x_n
y	y_o	y_1	y_2	. . .	y_n

Ta xay döng ña thöc noi suy ham $f(x)$ tren $[a,b]=[x_0, x_n]$.

Ñaït

$$p_n^{(k)}(x) = \frac{\prod_{i=0, i\neq k}^{n} (x - x_i)}{\prod_{i=0, i\neq k}^{n} (x_k - x_i)}$$

$$= \frac{(x - x_0)(x - x_1)\ldots(x - x_{k-1})(x - x_{k+1})\ldots(x - x_n)}{(x_k - x_0)(x_k - x_1)\ldots(x_k - x_{k-1})(x_k - x_{k+1})\ldots(x_k - x_n)}$$

Ta co

$$p_n^{(k)}(x_i) = \begin{cases} 1 & i = k \\ 0 & i \neq k \end{cases}$$

Ña thöc

$$L_n(x) = \sum_{k=0}^{n} p_n^{(k)}(x)\, y_k$$

co bac $\leq$ n va thoa ñieu kien $L_n(x_k) = y_k$

goi la ña thöc noi suy Lagrange cua ham f

Ví du : Cho ham f va bang so

x	0	1	3
y	1	-1	2

Xay döng ña thöc noi suy Lagrange va tính gan ñung f(2).

Giai

$n = 2$

$$p_n^{(0)}(x) = \frac{(x-1)(x-3)}{(0-1)(0-3)} = \frac{1}{3}(x^2 - 4x + 3)$$

$$p_n^{(1)}(x) = \frac{(x-0)(x-3)}{(1-0)(1-3)} = -\frac{1}{2}(x^2 - 3x)$$

$$p_n^{(2)}(x) = \frac{(x-0)(x-1)}{(3-0)(3-1)} = \frac{1}{6}(x^2 - x)$$

Ña thöc noi suy Lagrange

$$L_n(x) = \frac{1}{3}(x^2 - 4x + 3) + \frac{1}{2}(x^2 - 3x) + \frac{1}{3}(x^2 - x) = \frac{7}{6}x^2 - \frac{19}{6}x + 1$$

$f(2) \approx L_n(2) = -2/3$

❖ Caùch bieåu dieãn khaùc :

Ñaët $\omega(x) = (x- x_0)(x- x_1) \ldots. (x- x_n)$

$$\omega'(x) = \sum_{k=0}^{n} \prod_{\substack{i=0 \\ i \neq k}}^{n} (x - x_i)$$

$$\omega'(x_k) = (x_k-x_0)(x_k-x_1)\ldots(x_k-x_{k-1})(x_k-x_{k+1})\ldots(x_k- x_n)$$

$$\Rightarrow \quad p_n^{(k)}(x) = \frac{\omega(x)}{\omega'(x_k)(x - x_k)}$$

$$\Rightarrow \quad L_n(x) = \omega(x) \sum_{k=0}^{n} \frac{y_k}{\omega'(x_k)(x - x_k)}$$

$$\Rightarrow \quad \boxed{L_n(x) = \omega(x) \sum_{k=0}^{n} \frac{y_k}{D_k}}$$ vôùi $D_k = \omega'(x_k)\,(x-x_k)$

Ñe tính gia trò cua $L_n(x)$, ta lap bang

x	x_0	x_1	....	x_n	
x_0	$x - x_0$	$x_0 - x_1$	....	$x_0 - x_n$	D_0
x_1	$x_1 - x_0$	$x - x_1$	....	$x_1 - x_n$	D_1
…	....	....	....	....	…
x_n	$x_n - x_0$	$x_n - x_1$	....	$x - x_n$	D_n
					$\omega(x)$

tích dong

tích ñöông cheo

Ví du : Cho ham f va bang so

x	-9	-7	-4
y	-1	-4	-9

Tính gan ñung f(-6)

Ta lap bang tai x = -6

x = -6	-9	-7	-4	
-9	3	-2	-5	30
-7	2	1	-3	-6
-4	5	3	-2	-30
				-6

Vay $f(-6) \approx L_2(-6) = -6(-1/30+4/6+9/30) = -5.6$

Ví du : Cho ham f va bang so

x	0	1	3	4
y	1	1	2	-1

Tính gan ñung f(2)

Ta lap bang tai x = 2

x = 2	0	1	3	4	
0	2	-1	-3	-4	-24
1	1	1	-2	-3	6
3	3	2	-1	-1	6
4	4	3	1	-2	-24
					4

Vay $f(2) \approx L_3(2) = 4(-1/24 + 1/6 + 1/3 + 1/24) = 2$

- TH ñac biet : cac ñiem nut cach ñeu

vôi böôc $h = x_{k+1} - x_k$

Ñat $$q = \frac{(x - x_0)}{h}$$

Ta co $x_k = x_o + kh$

$$\Rightarrow \quad x-x_k = x- x_o-kh = (q-k)h$$

$$x_i-x_j = (x_o+ih)-(x_o+jh) = (i-j)h$$

$$\Rightarrow \quad \omega(x)=(x-x_0)(x-x_1) \dots (x-x_n)=q(q-1)\dots(q-n)h^{n+1}$$

$$\omega'(x_k) = (x_k-x_0) \dots (x_k-x_{k-1})(x_k-x_{k+1}) \dots (x_k-x_n)$$

$$= k.(k-1) \dots 1.(-1)(-2) \dots (k-n)h^n$$

$$= (-1)^{n-k}\, k!\,(n-k)!\, h^n$$

$$L_n(x) = \omega(x)\sum_{k=0}^{n}\frac{y_k}{\omega'(x_k)(x-x_k)}$$

$$\Rightarrow \boxed{L_n(x) = q(q-1)\ldots(q-n)\sum_{k=0}^{n}\frac{(-1)^{n-k}y_k}{k!(n-k)!(q-k)}}$$

Ví dụ : Cho hàm f và bảng số

x	1.1	1.2	1.3	1.4
y	15	18	19	24

Tính gần đúng f(1.25)

giai

Ta co n = 3 x = 1.25

h = 0.1 q = (1.25-1.1)/0.1 = 1.5

$$L_n(1.25) = (1.5)(0.5)(-0.5)(-1.5)[-\frac{15}{3!(1.5)} + \frac{18}{2!(0.5)} - \frac{19}{2!(-0.5)} + \frac{24}{3!(-1.5)}]$$

$$= 18.375$$

Vay $f(1.25) \approx 18.375$

❖ Cong thöc ñanh gia sai so :

Gia sö ham f(x) co ñao ham ñen cap n+1 lien tuc tren [a,b].

Ñat $$M_{n+1} = \max_{x\in[a,b]} | f^{(n+1)}(x) |$$

Ta co cong thöc sai so

$$\boxed{| f(x) - L_n(x) | \le \frac{M_{n+1}}{(n+1)!} | \omega(x) |}$$

Ví du : Cho ham $f(x)=2^x$ tren ñoan [0,1]. Ñanh gia sai so khi tính gan ñung gia trò ham tai ñiem $x=0.45$ sö dung ña thöc noi suy Lagrange khi chon cac ñiem nut $x_o=0$, $x_1=0.25$, $x_2=0.5$, $x_3=0.75$, $x_4=1$

Giai

Ta co $n = 4$, $f^{(5)}(x) = (\ln 2)^5 2^x$

$$\Rightarrow M_5 = \max |f^{(5)}(x)| = 2(\ln 2)^5$$

cong thöc sai so

$$|f(x) - L_n(x)| \le \frac{M_{n+1}}{(n+1)!} |\omega(x)|$$

$$= \frac{2(\ln 2)^5}{5!} |(0.45)(0.20)(-0.05)(-0.30)(-0.55)| = 0.198 x 10^{-5}$$

III. ÑA THÖÙC NOÄI SUY NEWTON:

1. Tæ sai phaân :

Cho haøm y = f(x) xaùc ñònh treân [a,b]=[x_o, x_n] vaø baûng soá

x	x_o	x_1	x_2	. . .	x_n
y	y_o	y_1	y_2	. . .	y_n

Ñaïi löôïng $$f[x_k, x_{k+1}] = \frac{f(x_{k+1}) - f(x_k)}{x_{k+1} - x_k}$$

goïi laø <u>tæ sai phaân caáp 1</u> cuûa haøm f treân [x_k,x_{k+1}]

Tæ sai phan cap 2

$$f[x_k, x_{k+1}, x_{k+2}] = \frac{f[x_{k+1}, x_{k+2}] - f[x_k, x_{k+1}]}{x_{k+2} - x_k}$$

Bang qui nap ta ñònh nghóa tæ sai phan cap p

$$f[x_k, x_{k+1}, \dots, x_{k+p}] = \frac{f[x_{k+1}, x_{k+2}, \dots, x_{k+p}] - f[x_k, x_{k+1}, \dots, x_{k+p-1}]}{x_{k+p} - x_k}$$

Ví dụ : Cho hàm f và bảng số

x	1.0	1.3	1.6	2.0
y	0.76	0.62	0.46	0.28

Tính các tỉ sai phân

Giải : ta lập bảng các tỉ sai phân

k	x_k	$f(x_k)$	$f[x_k,x_{k+1}]$	$f[x_k,x_{k+1},x_{k+2}]$	$f[x_k,x_{k+1},x_{k+2},x_{k+3}]$
0	1.0	0.76			
1	1.3	0.62			
2	1.6	0.46			
3	2.0	0.28			

2. Ña thöc noi suy Newton :

Tæ sai phan cap 1

$$f[x,x_0]=\frac{f(x_0)-f(x)}{x_0-x}$$

$$\Rightarrow\ f(x)=y_0+f[x,x_0](x-x_0)$$

Tæ sai phan cap 2

$$f[x,x_0,x_1]=\frac{f[x_0,x_1]-f[x,x_0]}{x_1-x}$$

$$\Rightarrow\ f[x,x_0]=f[x_0,x_1]+f[x,x_0,x_1](x-x_1)$$

nein $\quad f(x)=y_0+f[x_0,x_1](x-x_0)+f[x,x_0,x_1](x-x_0)(x-x_1)$

Tiep tuc bang qui nap ta ñöôc

$$f(x) = y_0 + f[x_0, x_1](x - x_0) + f[x_0, x_1, x_2](x - x_0)(x - x_1) + \ldots$$
$$+ f[x_0, x_1, \ldots, x_n](x - x_0)(x - x_1)\ldots(x - x_{n-1})$$
$$+ f[x, x_0, \ldots, x_n](x - x_0)(x - x_1)\ldots(x - x_n)$$

Ñat

$$\aleph_n^{(1)}(x) = y_0 + f[x_0, x_1](x - x_0) + f[x_0, x_1, x_2](x - x_0)(x - x_1) + \ldots$$
$$+ f[x_0, x_1, \ldots, x_n](x - x_0)(x - x_1)\ldots(x - x_{n-1})$$
$$\Re_n(x) = f[x, x_0, \ldots, x_n](x - x_0)(x - x_1)\ldots(x - x_n)$$

Ta ñöôc $$\boxed{f(x) = \aleph_n^{(1)}(x) + \Re_n(x)}$$

Cong thöc nay goi la cong thöc Newton tien xuat phat tö ñiem nut x_o

Töông tö ta co cong thöc Newton lui

$$\boxed{f(x) = \aleph_n^{(2)}(x) + \Re_n(x)}$$

$$\aleph_n^{(2)}(x) = y_n + f[x_{n-1}, x_n](x - x_n) + f[x_{n-2}, x_{n-1}, x_n](x - x_n)(x - x_{n-1}) + \ldots + f[x_0, x_1, \ldots, x_n](x - x_n)(x - x_{n-1})\ldots(x - x_1)$$

$\aleph_n^{(1)}(x)$: ña thöc noi suy Newton tien

$\aleph_n^{(2)}(x)$: ña thöc noi suy Newton lui

$\Re_n(x)$: xac ñònh sai so

Neu ham f co ñao ham lien tuc ñen cap n+1, ta co cong thöc ñanh gia sai so :

$$\boxed{|\Re_n(x)| \le \frac{M_{n+1}}{(n+1)!}|\omega(x)| \text{ vôi } M_{n+1} = \max|f^{(n+1)}(x)|}$$

Ví du : Cho ham f xac ñình tren [0,1] va bang so

x	0	0.3	0.7	1
y	2	2.2599	2.5238	2.7183

Tính gan ñung f(0.12) bang Newton tien va f(0.9) bang Newton lui

Giai : ta lap bang cac tæ sai phan

x_k	$f(x_k)$	$f[x_k,x_{k+1}]$	$f[x_k,x_{k+1},x_{k+2}]$	$f[x_k,x_{k+1},x_{k+2},x_{k+3}]$
0	2			
		0.8663		Newton tien
0.3	2.2599		-0.2950	
		0.6598		0.2786
0.7	2.5238		-0.0164	
		0.6483		Newton lui
1	2.7183			

Ta co

$$f(0.12) \approx \aleph_n^{(1)}(0.12)$$
$$= 2 + 0.8663(0.12) - 0.2950(0.12)(-0.18) + 0.2786(0.12)(-0.18)(-0.58)$$
$$= 2.1138$$

$$f(0.9) \approx \aleph_n^{(2)}(0.9)$$
$$= 2.7183 + 0.6483(-0.1) - 0.0164(-0.1)(0.2) + 0.2786(-0.1)(0.2)(0.6)$$
$$= 2.6505$$

3. TH caùc ñieåm nuùt caùch ñeàu :

Sai phaân höõu haïn caáp 1 cuûa haøm taïi ñieåm x_k

$$\Delta y_k = y_{k+1} - y_k$$

Baèng qui naïp, Sai phaân höõu haïn caáp p cuûa haøm taïi ñieåm x_k

$$\Delta^p y_k = \Delta(\Delta^{p-1} y_k) = \Delta^{p-1} y_{k+1} - \Delta^{p-1} y_k$$

Ta coù coâng thöùc

$$f[x_k, x_{k+1}, \ldots, x_{k+p}] = \frac{\Delta^p y_k}{p!\,h^p}$$

Cong thöc Newton tien

Ñat $q = \dfrac{(x - x_0)}{h}$

$$\aleph_n^{(1)}(x) = y_0 + f[x_0, x_1](x - x_0) + f[x_0, x_1, x_2](x - x_0)(x - x_1) + \ldots$$

$$+ f[x_0, x_1, \ldots, x_n](x - x_0)(x - x_1)\ldots(x - x_{n-1})$$

$$= y_0 + \frac{\Delta y_0}{1!} q + \frac{\Delta^2 y_0}{2!} q(q-1) + \ldots + \frac{\Delta^n y_0}{n!} q(q-1)\ldots(q-n+1)$$

Cong thöc Newton lui

Ñat $p = \dfrac{(x - x_n)}{h}$

$$\aleph_n^{(2)}(x) = y_n + \frac{\Delta y_{n-1}}{1!} p + \frac{\Delta^2 y_{n-2}}{2!} p(p+1) + \ldots + \frac{\Delta^n y_0}{n!} p(p+1)\ldots(p+n-1)$$

Ví du : Cho ham f xac ñònh va bang so

x	30	35	40	45
y	0.5	0.5736	0.6428	0.7071

Tính gan ñung f(32) va f(44)

Giai : ta lap bang cac sai phan höu han

x_k	$f(x_k)$	Δy_k	$\Delta^2 y_k$	$\Delta^3 y_k$
30	0.5			
		0.0736		Newton tien
35	0.5736		-0.0044	
		0.0692		-0.0005
40	0.6428		-0.0049	
		0.0643		Newton lui
45	0.7071			

- Tính gan ñung f(32) : dung cong thöc Newton tien

$$n = 3,\quad x_o = 30,\quad q=(32-30)/5 = 0.4$$

$$f(32) \approx \aleph_n^{(1)}(32)$$

$$= 0.5 + \frac{0.0736}{1!}(0.4) - \frac{0.0044}{2!}(0.4)(-0.6) - \frac{0.0005}{3!}(0.4)(-0.6)(-1.6)$$

$$= 0.529936$$

- Tính gan ñung f(44) : dung cong thöc Newton lui

$$n = 3,\quad x_n = 45,\quad p=(44-45)/5 = -0.2$$

$$f(44) \approx \aleph_n^{(2)}(44)$$

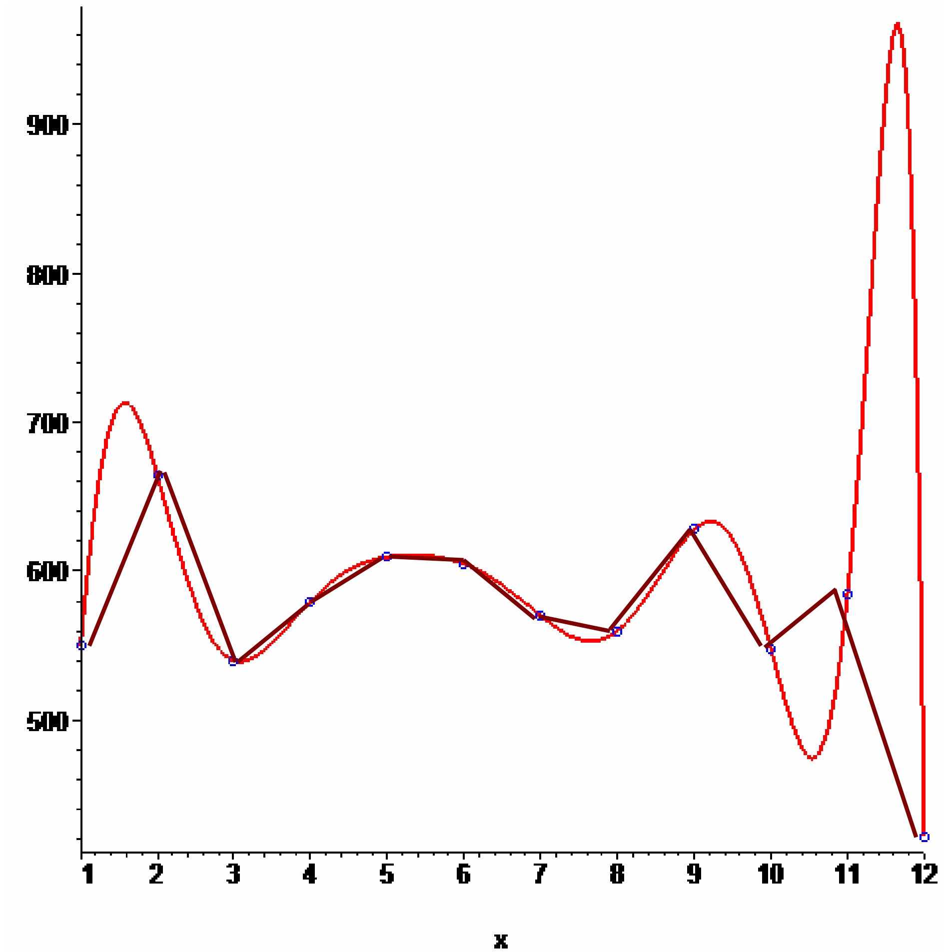
$$= 0.7071 + \frac{0.0643}{1!}(-0.2) - \frac{0.0049}{2!}(-0.2)(0.8) - \frac{0.0005}{3!}(-0.2)(0.8)(1.8)$$

$$= 0.694656$$

IV. SPLINE bac 3 :

Vôi n lôn, ña thöc noi suy bac rat lôn, khoi xay döng va khoi öng dung.

Mot cach khac phuc la thay ña thöc noi suy bac n bang cac ña thöc bac thap (≤ 3) tren töng ñoan $[x_k, x_{k+1}]$, $k=0,1,\ldots,n-1$



1. Ñònh nghóa :

Cho ham y=f(x) xac ñònh tren ñoan [a,b] va bang so

x	$a=x_o$	x_1	x_2	. . .	$x_n=b$
y	y_o	y_1	y_2	. . .	y_n

Mot Spline bac 3 noi suy ham f(x) la ham g(x) thoa cac ñieu kien sau :

(i) g(x) co ñao ham ñen cap 2 lien tuc tren [a,b]

(ii) $g(x)=g_k(x)$ la 1 ña thöc bac 3 tren $[x_k,x_{k+1}]$, k=0,1,..,n-1

(iii) $g(x_k) = y_k$, k=0,1, …, n

2. Cach xay döng Spline bac 3 :

Ñat $h_k = x_{k+1} - x_k$

$g_k(x)$ la ña thöc bac 3 nen co dang :

$$g_k(x) = a_k + b_k(x-x_k) + c_k(x-x_k)^2 + d_k(x-x_k)^3$$

- Ta co $g(x_k) = y_k$

$\Rightarrow \quad a_k = y_k,\ k = 0,1,\ldots,n$

- g(x) kha vi lien tuc ñen cap 2 nen

(A) $g_k(x_{k+1}) = g_{k+1}(x_{k+1})$

(B) $g'_k(x_{k+1}) = g'_{k+1}(x_{k+1}),\ \forall k = 0,1,..,n-1$

(C) $g''_k(x_{k+1}) = g''_{k+1}(x_{k+1})$

- Ñieu kien (A) suy ra

$$a_k + b_k h_k + c_k h_k^2 + d_k h_k^3 = y_{k+1}$$

$$\Rightarrow \qquad b_k = \frac{(y_{k+1} - y_k)}{h_k} - c_k h_k - d_k h_k^2 \qquad (1)$$

Ta co $\qquad g_k'(x) = b_k + 2c_k(x - x_k) + 3d_k(x - x_k)^2$

$$g_k''(x) = 2c_k + 6d_k(x - x_k)$$

- Ñieu kien (C) suy ra

$$2c_k + 6d_k h_k = 2c_{k+1}$$

$$\Rightarrow \qquad d_k = \frac{(c_{k+1} - c_k)}{3h_k} \qquad (2)$$

- Thay (2) vaøo (1) ta ñöôïc

$$b_k = \frac{(y_{k+1} - y_k)}{h_k} - \frac{(c_{k+1} + 2c_k)h_k}{3} \qquad (3)$$

- Ñieàu kieän (B) suy ra

$$b_k + 2c_k h_k + 3d_k h_k^2 = b_{k+1}$$

hay $$b_{k-1} + 2c_{k-1}h_{k-1} + 3d_{k-1}h_{k-1}^2 = b_k \qquad (4)$$

- Thay (2) vaø (3) vaøo (4) ta ñöôïc

$$h_{k-1}c_{k-1} + 2(h_{k-1} + h_k)c_k + h_k c_{k+1} = \frac{3(y_{k+1} - y_k)}{h_k} - \frac{3(y_k - y_{k-1})}{h_{k-1}} \qquad (5)$$

$$\forall k = 1, 2, \ldots, n-1$$

Phöông trình (5) la he pttt gom n-1 pt, dung ñe xac ñònh cac he so c_k. Töø c_k va (2) (3) ta xac ñònh ñöôc tat ca cac he so cua ña thöùc $g_k(x)$

Phöông trình (5) co vo so nghiem, ñe co nghiem duy nhat ta can bo sung them 1 so ñieu kien

❖ Ñònh nghóa :

- Spline töï nhien la spline vôùi ñieu kien

$$g''(a) = g''(b) = 0$$

- Spline rang buoc la spline vôùi ñieu kien

$$g'(a) = \alpha, \quad g'(b) = \beta$$

3. Spline tö nhien :

Giai thuat xac ñònh spline tö nhien :

Ñieu kien g"(a)=g"(b) = 0 suy ra $c_o = c_n = 0$

B1. Tính $h_k = x_{k+1} - x_k$, $k = 0, n-1$.

$a_k = y_k$, $k = 0, n$

B2. Giai he $Ac = b$ tìm $c = (c_o, c_1, \ldots, c_n)^t$

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 & \ldots & 0 \\ h_0 & 2(h_0 + h_1) & h_1 & 0 & \ldots & 0 \\ 0 & h_1 & 2(h_1 + h_2) & h_2 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & h_{n-2} & 2(h_{n-2} + h_{n-1}) & h_{n-1} \\ 0 & 0 & 0 & 0 & \ldots & 1 \end{pmatrix} \qquad b = \begin{pmatrix} 0 \\ \dfrac{3(y_2 - y_1)}{h_1} - \dfrac{3(y_1 - y_0)}{h_0} \\ \ldots \\ \dfrac{3(y_n - y_{n-1})}{h_{n-1}} - \dfrac{3(y_{n-1} - y_{n-2})}{h_{n-2}} \\ 0 \end{pmatrix}$$

B3. Tính các hệ số b_k, d_k.

$$b_k = \frac{(y_{k+1} - y_k)}{h_k} - \frac{(c_{k+1} + 2c_k)h_k}{3}, \ k = 0,1,\dots,n-1$$

$$d_k = \frac{(c_{k+1} - c_k)}{3h_k}$$

Ví dụ : Xây dựng spline tự nhiên nội suy hàm theo bảng số

x	0	2	5
y	1	1	4

Giải

$n = 2$

B1. $h_o = 2$, $h_1 = 3$. $a_o = 1$, $a_1 = 1$, $a_2 = 4$

B2. Giải hệ $Ac = b$ với $c = (c_0, c_1, c_2)^t$

$$A = \begin{pmatrix} 1 & 0 & 0 \\ h_0 & 2(h_0 + h_1) & h_1 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 10 & 3 \\ 0 & 0 & 1 \end{pmatrix}, \quad b = \begin{pmatrix} 0 \\ \dfrac{3(y_2 - y_1)}{h_1} - \dfrac{3(y_1 - y_0)}{h_0} \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 3 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 2 & 10 & 3 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} c_0 \\ c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 3 \\ 0 \end{pmatrix} \quad \Rightarrow c_o = c_2 = 0,\ c_1 = 3/10$$

B3. Tính cac he so b_k, d_k.

$$b_0 = \frac{(y_1 - y_0)}{h_0} - \frac{(c_1 + 2c_0)h_0}{3} = -\frac{1}{5}$$

$$b_1 = \frac{(y_2 - y_1)}{h_1} - \frac{(c_2 + 2c_1)h_1}{3} = \frac{2}{5}$$

$$d_0 = \frac{(c_1 - c_0)}{3h_0} = \frac{1}{20}, \quad d_1 = \frac{(c_2 - c_1)}{3h_1} = -\frac{1}{30}$$

Ket luan : spline tö nhien

$$g(x) = \begin{cases} g_0(x) = 1 - \frac{1}{5}x + \frac{1}{20}x^3 & 0 \le x \le 2 \\ g_1(x) = 1 + \frac{2}{5}(x-2) + \frac{3}{10}(x-2)^2 - \frac{1}{30}(x-2)^3 & 2 \le x \le 5 \end{cases}$$

Ví dui : Xaiy döing spline töi nhiein noii suy haim theo baing soi

x	0	1	2	3
y	1	2	4	8

$n = 3$

B1. $h_o = h_1 = h_2 = 1$. $a_o = 1$, $a_1 = 2$, $a_2 = 4$, $a_3 = 8$

B2. Giaii hei $Ac = b$ vôii $c = (c_0, c_1, c_2, c_3)^t$

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ h_0 & 2(h_0 + h_1) & h_1 & 0 \\ 0 & h_1 & 2(h_1 + h_2) & h_2 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 4 & 1 & 0 \\ 0 & 1 & 4 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$
b = \begin{pmatrix} 0 \\ \dfrac{3(y_2 - y_1)}{h_1} - \dfrac{3(y_1 - y_0)}{h_0} \\ \dfrac{3(y_3 - y_2)}{h_2} - \dfrac{3(y_2 - y_1)}{h_1} \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 3 \\ 6 \\ 0 \end{pmatrix}
$$

$$
\begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 4 & 1 & 0 \\ 0 & 1 & 4 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 3 \\ 6 \\ 0 \end{pmatrix} \Rightarrow \begin{cases} c_0 = c_3 = 0 \\ 4c_1 + c_2 = 3 \\ c_1 + 4c_2 = 6 \end{cases}
$$

Giai ta ñöôc $c_o = c_3 = 0$, $c_1 = 2/5$, $c_2 = 7/5$

B3. Tính các hệ số b_k, d_k.

$$b_0 = \frac{(y_1 - y_0)}{h_0} - \frac{(c_1 + 2c_0)h_0}{3} = \frac{13}{15}, \quad b_1 = \frac{(y_2 - y_1)}{h_1} - \frac{(c_2 + 2c_1)h_1}{3} = \frac{19}{15}$$

$$b_2 = \frac{(y_3 - y_2)}{h_2} - \frac{(c_3 + 2c_2)h_2}{3} = \frac{46}{15}$$

$$d_0 = \frac{(c_1 - c_0)}{3h_0} = \frac{2}{15}, \quad d_1 = \frac{(c_2 - c_1)}{3h_1} = \frac{1}{3}, \quad d_2 = \frac{(c_3 - c_2)}{3h_2} = -\frac{7}{15}$$

Kết luận : spline tự nhiên

$$g(x) = \begin{cases} g_0(x) = 1 + \dfrac{13}{15}x + \dfrac{2}{15}x^3 & 0 \le x \le 1 \\ g_1(x) = 2 + \dfrac{19}{15}(x-1) + \dfrac{2}{5}(x-1)^2 + \dfrac{1}{3}(x-1)^3 & 1 \le x \le 2 \\ g_2(x) = 4 + \dfrac{46}{15}(x-2) + \dfrac{7}{5}(x-2)^2 - \dfrac{7}{15}(x-2)^3 & 2 \le x \le 3 \end{cases}$$

4. Spline rang buoc :

Ñieu kien g'(a) = α, g'(b) = β xac ñònh 2 pt :

$$\begin{cases} 2h_0c_0 + h_0c_1 = 3\dfrac{y_1 - y_0}{h_0} - 3\alpha \\ h_{n-1}c_{n-1} + 2h_{n-1}c_n = 3\beta - 3\dfrac{y_n - y_{n-1}}{h_{n-1}} \end{cases}$$

Giai thuat xac ñònh spline rang buoc :

B1. Tính $h_k = x_{k+1} - x_k$, k = 0, n-1.

$a_k = y_k$, k = 0, n

B2. Giải hệ Ac = b tìm $c = (c_o, c_1, \ldots, c_n)^t$

$$A=\begin{pmatrix} 2h_0 & h_0 & 0 & 0 & \ldots & 0 \\ h_0 & 2(h_0+h_1) & h_1 & 0 & \ldots & 0 \\ 0 & h_1 & 2(h_1+h_2) & h_2 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & h_{n-2} & 2(h_{n-2}+h_{n-1}) & h_{n-1} \\ 0 & 0 & 0 & 0 & h_{n-1} & 2h_{n-1} \end{pmatrix}$$

$$b=\begin{pmatrix} 3\dfrac{y_1-y_0}{h_0}-3\alpha \\ \dfrac{3(y_2-y_1)}{h_1}-\dfrac{3(y_1-y_0)}{h_0} \\ \ldots \\ \dfrac{3(y_n-y_{n-1})}{h_{n-1}}-\dfrac{3(y_{n-1}-y_{n-2})}{h_{n-2}} \\ 3\beta-3\dfrac{y_n-y_{n-1}}{h_{n-1}} \end{pmatrix}$$

B3. Tính các hệ số b_k, d_k.

$$b_k=\frac{(y_{k+1}-y_k)}{h_k}-\frac{(c_{k+1}+2c_k)h_k}{3},\ k=0,1,\ldots,n-1$$

$$d_k=\frac{(c_{k+1}-c_k)}{3h_k}$$

Ví duï : Xaây döïng spline raøng buoäc noäi suy haøm theo baûng soá

x	0	1	2
y	1	2	1

vôùi ñieàu kieän $g'(0)=g'(2) = 0$

Giaûi

$n = 2$

B1. $h_o = h_1 = 1$. $a_o = 1$, $a_1 = 2$, $a_2 = 1$

B2. Giai he Ac = b vôi c = $(c_0, c_1, c_2)^t$

$$A = \begin{pmatrix} 2h_0 & h_0 & 0 \\ h_0 & 2(h_0 + h_1) & h_1 \\ 0 & h_1 & 2h_1 \end{pmatrix} = \begin{pmatrix} 2 & 1 & 0 \\ 1 & 4 & 1 \\ 0 & 1 & 2 \end{pmatrix}$$

$$b = \begin{pmatrix} 3\dfrac{y_1 - y_0}{h_0} - 3\alpha \\ \dfrac{3(y_2 - y_1)}{h_1} - \dfrac{3(y_1 - y_0)}{h_0} \\ 3\beta - 3\dfrac{y_2 - y_1}{h_1} \end{pmatrix} = \begin{pmatrix} 3 \\ -6 \\ 3 \end{pmatrix}$$

$$\begin{pmatrix} 2 & 1 & 0 \\ 1 & 4 & 1 \\ 0 & 1 & 2 \end{pmatrix}\begin{pmatrix} c_0 \\ c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 3 \\ -6 \\ 3 \end{pmatrix} \Rightarrow c_0 = 3, c_1 = -3, c_2 = 3$$

B3. Tính các hệ số b_k, d_k.

$$b_0 = \frac{(y_1 - y_0)}{h_0} - \frac{(c_1 + 2c_0)h_0}{3} = 0$$

$$b_1 = \frac{(y_2 - y_1)}{h_1} - \frac{(c_2 + 2c_1)h_1}{3} = 0$$

$$d_0 = \frac{(c_1 - c_0)}{3h_0} = -2, \quad d_1 = \frac{(c_2 - c_1)}{3h_1} = 2$$

Kết luận : spline ràng buộc

$$g(x) = \begin{cases} g_0(x) = 1 + 3x^2 - 2x^3 & 0 \le x \le 1 \\ g_1(x) = 2 - 3(x-1)^2 + 2(x-1)^3 & 1 \le x \le 2 \end{cases}$$

V. BAI TOAN XAP XÆ THÖC NGHIEM :

Xet bai toan thong ke löông möa trong 12 thang Thöc nghiem (k=1..12)

x_k	1	2	3	4	5	6	7	8
y_k	550	650	540	580	610	605	......	

Cac gia trò y_k ñöôc xac ñònh bang thöc nghiem nen co the khong chính xac. Khi ño viec xay döng mot ñöông cong ñi qua tat ca cac ñiem $M_k(x_k, y_k)$ cung khong con chính xac

Bai toan xap xæ thöïc nghiem : la tìm ham f(x) xap xæ bang $\{(x_k,y_k)\}$ theo phöông phap bình phöông cöïc tieu :

$$g(f) = \sum (f(x_k) - y_k)^2 \text{ ñat min}$$

Ham f tong quat rat ña dang. Ñe ñôn gian, trong thöïc te thöông ta tìm ham f theo mot trong cac dang sau :

- $f(x) = A + Bx$
- $f(x) = A+Bx+Cx^2$
- $f(x) = A\sin x+B\cos x$
- $f(x) = Ae^{Bx}$
- $f(x) = Ax^B$
- $f(x) = A\ln Bx$ …

1. Tröông hôp f(x) = A+ Bx :

Phöông trình bình phöông cöc tieu co dang

$$g(A,B) = \sum (A + Bx_k - y_k)^2$$

Bai toan qui ve tìm cöc tieu cua ham 2 bien g(A,B)

Ñiem döng

$$\begin{cases} \dfrac{\partial g}{\partial A} = 2\sum (A + Bx_k - y_k) = 0 \\ \dfrac{\partial g}{\partial B} = 2\sum (A + Bx_k - y_k)x_k = 0 \end{cases}$$

Suy ra

$$\begin{cases} nA + (\sum x_k)B = \sum y_k \\ (\sum x_k)A + (\sum x_k^2)B = \sum x_k y_k \end{cases}$$

Ví du : Tìm ham f(x) = A + Bx xap xæ bang so

x	1	1	2	2	2	3	3	4	5	6
y	1	2	2	3	4	4	5	5	6	7

Theo pp BPCT

Ta co n = 10

Giai he pt

$$\begin{cases} nA + (\sum x_k)B = \sum y_k \\ (\sum x_k)A + (\sum x_k^2)B = \sum x_k y_k \end{cases} \Rightarrow \begin{cases} 10A + 29B = 39 \\ 29A + 109B = 140 \end{cases}$$

Nghiem A = 0.7671, B=1.0803

Vay f(x) = 0.7671+1.0803x

2. Tröông hôp f(x) = Acosx + Bsinx :

Phöông trình bình phöông cöc tieu co dang

$$g(A,B)=\sum (A\cos x_k + B\sin x_k - y_k)^2$$

Bai toan qui ve tìm cöc tieu cua ham 2 bien g(A,B)

Ñiem döng
$$\begin{cases} \dfrac{\partial g}{\partial A} = 2\sum (A\cos x_k + B\sin x_k - y_k)\cos x_k = 0 \\ \dfrac{\partial g}{\partial B} = 2\sum (A\cos x_k + B\sin x_k - y_k)\sin x_k = 0 \end{cases}$$

Suy ra
$$\begin{cases} (\sum \cos^2 x_k)A + (\sum \sin x_k \cos x_k)B = \sum y_k \cos x_k \\ (\sum \sin x_k \cos x_k)A + (\sum \sin^2 x_k)B = \sum y_k \sin x_k \end{cases}$$

Ví dụ : Tìm hàm f(x)=Acosx+Bsinx xấp xỉ bằng số

x	10	20	30	40	50	rad
y	1.45	1.12	0.83	1.26	1.14	

Theo pp BPCT

Ta có n = 5

Giải hệ pt

$$\begin{cases} \left(\sum \cos^2 x_k\right)A + \left(\sum \sin x_k \cos x_k\right)B = \sum y_k \cos x_k \\ \left(\sum \sin x_k \cos x_k\right)A + \left(\sum \sin^2 x_k\right)B = \sum y_k \sin x_k \end{cases}$$

$$\Rightarrow \begin{cases} 2.2703A - 0.0735B = -0.3719 \\ -0.0735A + 2.7297B = 0.0533 \end{cases}$$

Nghiệm A = -0.1633, B=0.0151

Vậy f(x) = -0.1633cosx+0.0151sinx

3. Tröông hôp $f(x) = Ax^2 + B\sin x$:

Phöông trình bình phöông cöc tieu co dang

$$g(A,B) = \sum (Ax_k^2 + B\sin x_k - y_k)^2$$

Bai toan qui ve tìm cöc tieu cua ham 2 bien g(A,B)

Ñiem döng

$$\begin{cases} \dfrac{\partial g}{\partial A} = 2\sum (Ax_k^2 + B\sin x_k - y_k)x_k^2 = 0 \\ \dfrac{\partial g}{\partial B} = 2\sum (Ax_k^2 + B\sin x_k - y_k)\sin x_k = 0 \end{cases}$$

Suy ra

$$\begin{cases} (\sum x_k^4)A + (\sum x_k^2 \sin x_k)B = \sum x_k^2 y_k \\ (\sum x_k^2 \sin x_k)A + (\sum \sin^2 x_k)B = \sum y_k \sin x_k \end{cases}$$

Ví dụ : Tìm hàm $f(x)=Ax^2+B\sin x$ xấp xỉ bảng số

x	1.3	1.5	1.8	2.0	2.4	2.6	2.7
y	2.7	1.8	3.51	3.1	3.78	3.9	4.32

Theo pp BPCT

Ta có n =7

Giải hệ pt

$$\begin{cases} (\sum x_k^4)A + (\sum x_k^2 \sin x_k)B = \sum x_k^2 y_k \\ (\sum x_k^2 \sin x_k)A + (\sum \sin^2 x_k)B = \sum y_k \sin x_k \end{cases}$$

$$\Rightarrow \begin{cases} 166.4355A + 21.1563B = 112.015 \\ 21.1563A + 4.6033B = 17.0441 \end{cases}$$

Nghiệm A = 0.4867, B=1.4657

Vậy $f(x) = 0.4857x^2 + 1.4657\sin x$

4. Tröông hôp $f(x) = A + Bx + Cx^2$:

Phöông trình bình phöông cöc tieu co dang

$$g(A,B,C) = \sum (A + Bx_k + Cx_k^2 - y_k)^2$$

Bai toan qui ve tìm cöc tieu cua ham 3 bien g(A,B,C)

Ñiem döng

$$\begin{cases} \dfrac{\partial g}{\partial A} = 2\sum (A + Bx_k + Cx_k^2 - y_k) = 0 \\ \dfrac{\partial g}{\partial B} = 2\sum (A + Bx_k + Cx_k^2 - y_k)x_k = 0 \\ \dfrac{\partial g}{\partial C} = 2\sum (A + Bx_k + Cx_k^2 - y_k)x_k^2 = 0 \end{cases}$$

Suy ra

$$\begin{cases} nA + (\sum x_k)B + (\sum x_k^2)C = \sum y_k \\ (\sum x_k)A + (\sum x_k^2)B + (\sum x_k^3)C = \sum x_k y_k \\ (\sum x_k^2)A + (\sum x_k^3)B + (\sum x_k^4)C = \sum x_k^2 y_k \end{cases}$$

Ví dụ : Tìm hàm $f(x) = A + Bx+Cx^2$ xấp xỉ bảng số

x	1	1	2	3	3	4	5
y	4.12	4.18	6.23	8.34	8.38	12.13	18.32

Theo pp BPCT

Ta có n = 7

Giải hệ pt

$$\begin{cases} nA + (\sum x_k)B + (\sum x_k^2)C = \sum y_k \\ (\sum x_k)A + (\sum x_k^2)B + (\sum x_k^3)C = \sum x_k y_k \\ (\sum x_k^2)A + (\sum x_k^3)B + (\sum x_k^4)C = \sum x_k^2 y_k \end{cases} \Rightarrow \begin{cases} 7A + 19B + 65C = 61.70 \\ 19A + 65B + 253C = 211.04 \\ 65A + 253B + 1061C = 835.78 \end{cases}$$

Nghiệm A = 4.3, B=-0.71, C=0.69

Vậy $f(x) = 4.3-0.71x+0.69x^2$