CHÖÔNG 3

HEÄ PHÖÔNG TRÌNH TUYEÁN TÍNH

I. ÑAT BAI TOAN :

He phöông trình tuyen tính n pt va n an co dang

$$Ax = b$$

vôi

$$A = (a_{ij}) = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix} \quad x = \begin{pmatrix} x_1 \\ x_2 \\ \dots \\ x_n \end{pmatrix} \quad b = \begin{pmatrix} b_1 \\ b_2 \\ \dots \\ b_n \end{pmatrix}$$

Cac phöông phap giai

- Phöông phap giai chính xac
  - Phöông phap Gauss
  - Phöông phap Gauss-Jordan
  - Phöông phap nhan töi LU
  - Phöông phap Cholesky
- Phöông phap giai gan ñung
  - Phöông phap lap Jacobi
  - Phöông phap lap Gauss-Seidel

II. PHÖÔNG PHAÙP GAUSS

1. Caùc daïng ma traän ñaëc bieät :

a. Ma traän cheùo :

$$A = \begin{pmatrix} a_{11} & 0 & \dots & 0 \\ 0 & a_{22} & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & a_{nn} \end{pmatrix}$$

$\det A = a_{11}a_{22} \dots a_{nn} \neq 0 \Leftrightarrow a_{ii} \neq 0, \forall i$

Nghieäm $x_i = b_i/a_{ii}$

b. Ma traän tam giaùc döôùi

$$A=\begin{pmatrix} a_{11} & 0 & \dots & 0 \\ a_{21} & a_{22} & \dots & 0 \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix}$$

$\det A = a_{11}a_{22} \dots a_{nn} \neq 0 \Leftrightarrow a_{ii} \neq 0, \forall i$

Phöông trình coù nghieäm

$$\begin{cases} x_1 = \dfrac{b_1}{a_{11}} \\ x_k = \dfrac{1}{a_{kk}}[b_k - \sum_{j=1}^{k-1} a_{kj}x_j], k = 2, n \end{cases}$$

c. Ma traän tam giaùc treân :

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ 0 & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & a_{nn} \end{pmatrix}$$

$\det A = a_{11}a_{22} \ldots a_{nn} \neq 0 \Leftrightarrow a_{ii} \neq 0, \forall i$

Phöông trình coù nghieäm

$$\begin{cases} x_n = \dfrac{b_n}{a_{nn}} \\ x_k = \dfrac{1}{a_{kk}}[b_k - \displaystyle\sum_{j=k+1}^{n} a_{kj}x_j], k = \overline{1, n-1} \end{cases}$$

2. Phöông phap Gauss :

Ta sö dung cac phep bien ñoi sô cap theo dong ñe chuyen ma tran A ve ma tran tam giac tren

Cac phep bien ñoi sô cap theo dong

- hoan chuyen 2 dong
- nhan 1 dong vôi 1 so khac 0
- cong 1 dong vôi dong khac

Ví du : Giai he phöông trình

$$\begin{cases} x_1 - x_2 + 2x_3 - x_4 = -8 \\ 2x_1 - 2x_2 + 3x_3 - 3x_4 = -20 \\ x_1 + x_2 + x_3 = -2 \\ x_1 - x_2 + 4x_3 + 3x_4 = 4 \end{cases}$$

Giai

$$[A/b] = \left(\begin{array}{cccc|c} 1 & -1 & 2 & -1 & -8 \\ 2 & -2 & 3 & -3 & -20 \\ 1 & 1 & 1 & 0 & -2 \\ 1 & -1 & 4 & 3 & 4 \end{array}\right) \xrightarrow[h_4=h_4-h_1]{\substack{h_2=h_2-2h_1 \\ h_3=h_3-h_1}} \left(\begin{array}{cccc|c} 1 & -1 & 2 & -1 & -8 \\ 0 & 0 & -1 & -1 & -4 \\ 0 & 2 & -1 & 1 & 6 \\ 0 & 0 & 2 & 4 & 12 \end{array}\right)$$

$$\xrightarrow{\substack{h_2 \leftrightarrow h_3 \\ h_4=h_4/2}} \left(\begin{array}{cccc|c} 1 & -1 & 2 & -1 & -8 \\ 0 & 2 & -1 & 1 & 6 \\ 0 & 0 & -1 & -1 & -4 \\ 0 & 0 & 1 & 2 & 6 \end{array}\right) \xrightarrow{h4=h4+h3} \left(\begin{array}{cccc|c} 1 & -1 & 2 & -1 & -8 \\ 0 & 2 & -1 & 1 & 6 \\ 0 & 0 & -1 & -1 & -4 \\ 0 & 0 & 0 & 1 & 2 \end{array}\right)$$

Giai pt ma tran tam giac tren, ta ñööc nghiem

$$x = (-7, 3, 2, 2)^t$$

III. PHÖÔNG PHAP NHAN TÖ LU

Phan tích ma tran A thanh tích 2 ma tran L va U

A = LU

L : ma tran tam giac döôi

U : ma tran tam giac tren

Phöông trình Ax = b $\Leftrightarrow$ L(Ux) = b

Ta ñöa ve giai 2 he phöông trình

$$\begin{cases} Ly = b \\ Ux = y \end{cases}$$

Phöông phap Doolittle :

Gia sö A ma tran khong suy bien va $a_{11} \neq 0$

Ta co the phan tích A thanh

$$A = LU$$

$$L = \begin{pmatrix} 1 & 0 & \dots & 0 \\ l_{21} & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ l_{n1} & l_{n2} & \dots & 1 \end{pmatrix}$$

Ma tran Δ döôi

$$U = \begin{pmatrix} u_{11} & u_{12} & \dots & u_{1n} \\ 0 & u_{22} & \dots & u_{2n} \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & u_{nn} \end{pmatrix}$$

Ma tran Δ tren

Cac phan tö cua L va U ñöôc xac ñònh theo cong thöc

$$\begin{cases} u_{1j} = a_{1j}, \quad 1 \le j \le n \\ l_{i1} = \dfrac{a_{i1}}{u_{11}}, \quad 2 \le i \le n \\ u_{ij} = a_{ij} - \sum\limits_{k=1}^{i-1} l_{ik}u_{kj}, \quad 1 < i \le j \\ l_{ij} = \dfrac{1}{u_{jj}}[a_{ij} - \sum\limits_{k=1}^{j-1} l_{ik}u_{kj}], \quad 1 < j < i \end{cases}$$

Ví du : Giai he phöông trình

$$\begin{cases} 2x_1 + 2x_2 - 3x_3 = 9 \\ -4x_1 - 3x_2 + 4x_3 = -15 \\ 2x_1 + x_2 + 2x_3 = 3 \end{cases}$$

Giai

Ta phan tích

$$A = \begin{pmatrix} 2 & 2 & -3 \\ -4 & -3 & 4 \\ 2 & 1 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 1 & l_{32} & 1 \end{pmatrix} \begin{pmatrix} 2 & 2 & -3 \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{pmatrix}$$

$$u_{22} = a_{22} - l_{21}u_{12} = 1$$

$$u_{23} = a_{23} - l_{21}u_{13} = -2$$

$$l_{32} = \frac{1}{u_{22}}(a_{32} - l_{31}u_{12}) = -1$$

$$u_{33} = a_{33} - l_{31}u_{13} - l_{32}u_{23} = 3$$

Giai he Ly = b

$$\begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 1 & -1 & 1 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} 9 \\ -15 \\ 3 \end{pmatrix} \Rightarrow y = \begin{pmatrix} 9 \\ 3 \\ -3 \end{pmatrix}$$

Giai he Ux = y

$$\begin{pmatrix} 2 & 2 & -3 \\ 0 & 1 & -2 \\ 0 & 0 & 3 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 9 \\ 3 \\ -3 \end{pmatrix} \Rightarrow x = \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}$$

Nghiem $x_1 = 2$, $x_2 = 1$, $x_3 = -1$

TH ñac biet : A ma tran 3 ñöông cheo

$$A = \begin{pmatrix} a_{11} & a_{12} & 0 & \dots & 0 & 0 \\ a_{21} & a_{22} & a_{23} & \dots & 0 & 0 \\ 0 & a_{32} & a_{33} & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & a_{nn-1} & a_{nn} \end{pmatrix}$$

Ta phan tích A thanh LU vôi

$$L = \begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ l_{21} & 1 & 0 & \dots & 0 \\ 0 & l_{32} & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 1 \end{pmatrix} \quad U = \begin{pmatrix} u_{11} & u_{12} & 0 & \dots & 0 \\ 0 & u_{22} & u_{23} & \dots & 0 \\ 0 & 0 & u_{33} & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & u_{nn} \end{pmatrix}$$

Cac phan tö cua L va U ñöôc xac ñònh theo cong thöc

$$
\begin{cases}
u_{11} = a_{11},\, u_{12} = a_{12},\, l_{21} = \dfrac{a_{21}}{u_{11}} \\
u_{ii} = a_{ii} - l_{i\,i-1} u_{i-1\,i}, \quad i = 2,n \\
u_{i\,i+1} = a_{i\,i+1}, \quad i = 2,n-1 \\
l_{i+1\,i} = \dfrac{a_{i+1\,i}}{u_{ii}}, \quad i = 2,n-1
\end{cases}
$$

Ví du : Giai he phöông trình Ax = b

$$A = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix} \quad b = \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}$$

Giai

Ta phan tích

$$A = \begin{pmatrix} 1 & 0 & 0 \\ -1/2 & 1 & 0 \\ 0 & l_{32} & 1 \end{pmatrix} \begin{pmatrix} 2 & -1 & 0 \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{pmatrix}$$

$$u_{22} = a_{22} - l_{21}u_{12} = 3/2$$

$$u_{23} = a_{23} = -1, \quad l_{32} = \frac{a_{32}}{u_{22}} = -2/3$$

$$u_{33} = a_{33} - l_{32}u_{23} = 4/3$$

Giải hệ Ly = b

$$\begin{pmatrix} 1 & 0 & 0 \\ -1/2 & 1 & 0 \\ 0 & -2/3 & 1 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix} \Rightarrow y = \begin{pmatrix} 2 \\ 2 \\ 10/3 \end{pmatrix}$$

Giải hệ Ux = y

$$\begin{pmatrix} 2 & -1 & 0 \\ 0 & 3/2 & -1 \\ 0 & 0 & 4/3 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \\ 10/3 \end{pmatrix} \Rightarrow x = \begin{pmatrix} 5/2 \\ 3 \\ 5/2 \end{pmatrix}$$

Nghiệm $x_1 = 5/2$, $x_2 = 3$, $x_3 = 5/2$

III. PHÖÔNG PHAP CHOLESKY

Ñònh nghóa :

- Ma tran A goi la ñoi xöng neu

$$A = A^t$$

- Ma tran A goi la xac ñònh döông neu

$$x^t A x = \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij} x_i x_j > 0, \forall x = (x_1, x_2, ..., x_n)^t \in R^n, x \neq 0$$

Ñe kiem tra xac ñònh döông, ta dung ñình ly sau:

Ñònh ly :

Ma tran A la xac ñònh döông khi va chæ khi tat ca cac ñònh thöc con chính cua no ñeu döông

Ví du : Kiem tra tính xac ñònh döông cua ma tran

$$A = \begin{pmatrix} 1 & 1 & -1 \\ 1 & 2 & 0 \\ -1 & 0 & 4 \end{pmatrix}$$

Giai

Cac ñònh thöc con chính: $\Delta_1 = 1 > 0, \Delta_2 = \begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix} = 1 > 0$

$$\Delta_3 = \begin{vmatrix} 1 & 1 & -1 \\ 1 & 2 & 0 \\ -1 & 0 & 4 \end{vmatrix} = -1\begin{vmatrix} 1 & 2 \\ -1 & 0 \end{vmatrix} - 0\begin{vmatrix} 1 & 1 \\ -1 & 0 \end{vmatrix} + 4\begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix} = 2 > 0$$

Vay A la xac ñònh döông

Ñònh lyù (Cholesky) :

Neáu A ma traän <u>ñoái xöùng vaø xaùc ñònh döông</u>, thì toàn taïi ma traän Δ döôùi, khaû ñaûo B sao cho

$$A = BB^t$$

Ma traän B = (b_{ij}) tìm theo coâng thöùc sau :

$$\begin{cases} b_{11} = \sqrt{a_{11}} \\ b_{i1} = \dfrac{a_{i1}}{b_{11}}, \ 2 \le i \le n \\ b_{ii} = \sqrt{a_{ii} - \displaystyle\sum_{k=1}^{i-1} b_{ik}^2}, \ 2 \le i \le n \\ b_{ij} = \dfrac{1}{b_{jj}}[a_{ij} - \displaystyle\sum_{k=1}^{j-1} b_{ik} b_{jk}], \ 2 \le j \le i \end{cases}$$

Ví duï : Giaûi heä phöông trình Ax = b

$$A = \begin{pmatrix} 1 & 1 & -1 \\ 1 & 2 & 0 \\ -1 & 0 & 4 \end{pmatrix} \qquad b = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

Giaûi

Ta coù A ma traän ñoái xöùng vaø xaùc ñònh döông

Phaân tích $A = BB^t$

$$B = \begin{pmatrix} 1 & 0 & 0 \\ 1 & b_{22} & 0 \\ -1 & b_{32} & b_{33} \end{pmatrix}$$

Caùc heä soá

$$\begin{cases} b_{22} = \sqrt{a_{22} - b_{21}^2} = 1 \\ b_{32} = \dfrac{1}{b_{22}}[a_{32} - b_{31}b_{21}] = 1 \\ b_{33} = \sqrt{a_{33} - b_{31}^2 - b_{32}^2} = \sqrt{2} \end{cases}$$

Giaỉi heä By = b

$$\begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ -1 & 1 & \sqrt{2} \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \Rightarrow y = \begin{pmatrix} 1 \\ 1 \\ 3/\sqrt{2} \end{pmatrix}$$

Giaỉi heä $B^t x = y$

$$\begin{pmatrix} 1 & 1 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & \sqrt{2} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 3/\sqrt{2} \end{pmatrix} \Rightarrow x = \begin{pmatrix} 3 \\ -1/2 \\ 3/2 \end{pmatrix}$$

Nghieäm $x_1 = 3$, $x_2 = -1/2$, $x_3 = 3/2$

IV. PHÖÔNG PHAP LAP

1. Chuan :

a. Chuan vector :

Ñònh nghóa :

Chuan cua vector $x \in R^n$ la ham so thöc ky hieu la $||x||$, thoa 3 ñieu kien sau :

(i) $||x|| \geq 0, \ \forall x \in R^n$ va $||x|| = 0 \Leftrightarrow x=0$

(ii) $||\lambda x|| = |\lambda| \ ||x||, \ \forall x \in R^n, \ \forall \ \lambda \in R$

(iii) $||x+y|| \leq ||x|| + ||y||, \ \forall x,y \in R^n$

Co nhieu cong thöc chuan khac nhau, xet 2 cong thöc

$\forall x = (x_1, x_2, \ldots, x_n)^t$

$$\| x \|_\infty = \max_{1 \le x \le n} \{ | x_i | \}$$

$$\| x \|_1 = \sum_{i=1}^{n} | x_i |$$

De dang kiem tra $\|x\|_\infty$, $\|x\|_1$ la cac chuan goi la chuan ∞ va chuan 1

b. Chuan ma tran :

Ñònh nghóa :

Chuan cua ma tran A ñöôc xac ñònh theo cong thöc

$$\| A \| = \max_{x \neq 0} \frac{\| Ax \|}{\| x \|} = \max_{\|x\|=1} \| Ax \|$$

Ñònh ly : Cho ma tran A = (a_{ij}), ta co

$$\| A \|_{\infty} = \max_{1 \leq i \leq n} \{ \sum_{j=1}^{n} | a_{ij} | \}$$

$$\| A \|_{1} = \max_{1 \leq j \leq n} \{ \sum_{i=1}^{n} | a_{ij} | \}$$

c. Hoi tu theo chuan :

Ñònh nghóa :

Day cac vector $\{x^{(m)}\} \in R^n$ hoi tu ve x theo chuan neu $\|x^{(m)} - x\| \to 0$ khi $m \to \infty$

Ñònh ly :

Day $\{x^{(m)} = (x_1^{(m)}, x_2^{(m)}, \ldots, x_n^{(m)})\} \in R^n$ hoi tu ve $x = (x_1, x_2, \ldots, x_n)$ theo chuan neu va chæ neu day $\{x_k^{(m)}\}$ hoi tu ve x_k khi $m \to \infty$, $\forall k = 1,n$

2. Phöông phap lap :

Ta chuyen he pt ve dang

$$x = Tx + c$$

Vôi T la ma tran vuong cap n va c la 1 vector

Ñe tìm nghiem gan ñung, vôi vector ban ñau $x^{(0)}$, ta xay döng day lap theo cong thöc

$$x^{(m)} = Tx^{(m-1)} + c, \ \forall m=1,2,\ldots$$

Ta can khao sat söi hoi tui cua day $\{x^{(m)}\}$

Ta co ñònh ly sau

Ñònh ly :

Neu ||T|| < 1 thì day lap $x^{(m)}$ se hoi tu ve nghiem x cua he pt, vôi moi vector ban ñau $x^{(0)}$.

Ta co cong thöc ñanh gia sai so :

$$(1) \quad \|x^{(m)} - x\| \le \frac{\|T\|^m}{1 - \|T\|}\|x^{(1)} - x^{(0)}\| \quad \text{tien nghiem}$$

$$\text{hoac} \quad (2) \quad \|x^{(m)} - x\| \le \frac{\|T\|}{1 - \|T\|}\|x^{(m)} - x^{(m-1)}\| \quad \text{hau nghiem}$$

V. PHÖÔNG PHAP LAP JACOBI

Ta phan tích

$$A = D - L - U$$

trong ño

$$D = \begin{pmatrix} a_{11} & 0 & \dots & 0 \\ 0 & a_{22} & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & a_{nn} \end{pmatrix}$$ ma tran cheo

$$L = \begin{pmatrix} 0 & 0 & \dots & 0 \\ -a_{21} & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ -a_{n1} & -a_{n2} & \dots & 0 \end{pmatrix}$$ ma tran Δ döôi

$$U = \begin{pmatrix} 0 & -a_{12} & \dots & -a_{1n} \\ 0 & 0 & \dots & -a_{2n} \\ \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 \end{pmatrix}$$ ma tran Δ tren

Phöông trình $Ax = b$

$$\Leftrightarrow \quad (D-L-U)x = b$$

$$\Leftrightarrow \quad Dx = (L+U)x + b$$

$$\Leftrightarrow \quad x = D^{-1}(L+U)x + D^{-1}b$$

$$\Leftrightarrow \quad x = Tx + c$$

vôi $T = D^{-1}(L+U)$ va $c = D^{-1}b$

pp lap theo phan tích tren goi la <u>pp lap Jacobi</u>

Bay giô ta tìm ñieu kien ñe pp lap Jacobi HT

Ñònh nghóa :

Ma traän A goïi laø ma traän ñöôøng cheùo troäi nghieâm ngaët neáu noù thoûa ñieàu kieän sau :

$$\sum_{j=1,\, j\neq i}^{n} |a_{ij}| < |a_{ii}|, \quad \forall i = \overline{1,n}$$

Nhaän xeùt :

Neáu A laø ma traän ñöôøng cheùo troäi nghieâm ngaët thì

$\det A \neq 0$ vaø $a_{ii} \neq 0 \ \forall i=\overline{1,n}$

Ñònh ly :

Neu A la ma tran ñöông cheo troi nghiem ngat, thì pp lap Jacobi hoi tu vôi moi gia trò ban ñau $x^{(0)}$

Ta co cong thöc lap Jacobi

$$x^{(m)} = Tx^{(m-1)} + c, \ \forall m=1,2,\dots$$

$$x_i^{(m)} = \frac{1}{a_{ii}}\left[-\sum_{j=1}^{i-1} a_{ij} x_i^{(m-1)} - \sum_{j=i+1}^{n} a_{ij} x_i^{(m-1)} + b_i\right], \ \forall i = \overline{1,n}$$

CM

Ta co $x^{(m)} = Tx^{(m-1)} + c$, $T = D^{-1}(L+U)$ va $c = D^{-1}b$

$$D^{-1} = \begin{pmatrix} 1/a_{11} & 0 & \dots & 0 \\ 0 & 1/a_{22} & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 1/a_{nn} \end{pmatrix} \quad T = \begin{pmatrix} 0 & -\frac{a_{12}}{a_{11}} & \dots & -\frac{a_{1n}}{a_{11}} \\ -\frac{a_{21}}{a_{22}} & 0 & \dots & -\frac{a_{2n}}{a_{22}} \\ \dots & \dots & \dots & \dots \\ -\frac{a_{n1}}{a_{nn}} & -\frac{a_{n2}}{a_{nn}} & \dots & 0 \end{pmatrix} \quad C = \begin{pmatrix} \frac{b_1}{a_{11}} \\ \frac{b_2}{a_{22}} \\ .. \\ \frac{b_n}{a_{nn}} \end{pmatrix}$$

A ma tran ñöông cheo troi nghiem ngat nen

$$\|T\|_\infty = \max_{1\le i\le n}\{\sum_{j=1, j\ne i}^{n} \frac{|a_{ij}|}{|a_{ii}|}\} < 1$$

$\Rightarrow$ pp lap hoi tu

Ví duï : Cho heä phöông trình

$$\begin{cases} 10x_1 + x_2 - x_3 = 7 \\ x_1 + 10x_2 + x_3 = 8 \\ -x_1 + x_2 + 10x_3 = 9 \end{cases}$$

a. Tìm nghieäm gaàn ñuùng $x^{(5)}$ vôùi vector ban ñaàu $x^{(0)} = 0$

b. Tính ma traän T vaø c

c. Tính sai soá cuûa nghieäm $x^{(5)}$

a.

$$A = \begin{pmatrix} 10 & 1 & -1 \\ 1 & 10 & 1 \\ -1 & 1 & 10 \end{pmatrix}$$ ma tran ñöông cheo troi nghiem ngat

Cong thöc lap Jacobi

$$\begin{cases} x_1^{(m)} = \dfrac{1}{10}(\qquad - x_2^{(m-1)} + x_3^{(m-1)} + 7) \\ x_2^{(m)} = \dfrac{1}{10}(-x_1^{(m-1)} \qquad - x_3^{(m-1)} + 8) \\ x_3^{(m)} = \dfrac{1}{10}(x_1^{(m-1)} - x_2^{(m-1)} \qquad + 9) \end{cases}$$

m	0	1	2	3	4	5
$x_1^{(m)}$	0					
$x_2^{(m)}$	0					
$x_3^{(m)}$	0					

b. Ta co

$$T = \begin{pmatrix} 0 & -0.1 & 0.1 \\ -0.1 & 0 & -0.1 \\ 0.1 & -0.1 & 0 \end{pmatrix} \quad c = \begin{pmatrix} 0.7 \\ 0.8 \\ 0.9 \end{pmatrix}$$

c. Cong thöc sai so

$$\| x^{(5)} - x \| \leq \frac{\| T \|}{1 - \| T \|} \| x^{(5)} - x^{(4)} \|$$

Ta co $\|T\|_{\infty}=0.2$, nen

$$\| x^{(5)} - x \| \leq \frac{0.2}{0.8} 4.9 * 10^{-4} = 0.1225 * 10^{-3}$$

VI. Phöông phaùp laëp Gauss-Seidel :

Ta phaân tích

$$A = D - L – U$$

nhö trong phaàn tröôùc

Phöông trình $Ax = b$

$$\Leftrightarrow \quad (D-L-U)x = b$$

$$\Leftrightarrow \quad (D-L)x = Ux + b$$

$$\Leftrightarrow \quad x = (D-L)^{-1}Ux + (D-L)^{-1}b$$

$$\Leftrightarrow \quad x = Tx + c$$

vôùi $T = (D-L)^{-1}U$ vaø $c = (D-L)^{-1}b$

pp laëp theo phaân tích naøy goïi laø <u>pp laëp Gauss-Seidel</u>

Ñònh ly :

Neu A la ma tran ñöông cheo troi nghiem ngat, thì pp lap Gauss-Seidel hoi tu vôi moi gia trò ban ñau $x^{(0)}$

Ta co cong thöc lap Gauss-Seidel

$$x^{(m)} = Tx^{(m-1)} + c,\ \forall m=1,2,\ldots$$

$$x_i^{(m)} = \frac{1}{a_{ii}}\left[-\sum_{j=1}^{i-1} a_{ij} x_i^{(m)} - \sum_{j=i+1}^{n} a_{ij} x_i^{(m-1)} + b_i\right],\ \forall i = \overline{1,n}$$

Ví du : Cho he phöông trình

$$\begin{cases} 20x_1 - x_2 + 2x_3 = 12 \\ x_1 + 20x_2 - x_3 = 13 \\ -2x_1 - x_2 + 20x_3 = 14 \end{cases}$$

a. Tìm nghiem gan ñung $x^{(4)}$ vôi vector ban ñau $x^{(0)} = 0$

b. Tính ma tran T va c

c. Tính sai so cua nghiem $x^{(4)}$

a.

$$A = \begin{pmatrix} 20 & -1 & 2 \\ 1 & 20 & -1 \\ -2 & -1 & 20 \end{pmatrix}$$ ma tran ñöông cheo troi nghiem ngat

Cong thöc lap Gauss-Seidel

$$\begin{cases} x_1^{(m)} = \dfrac{1}{20}(\qquad\qquad + x_2^{(m-1)} - 2x_3^{(m-1)} + 12) \\ x_2^{(m)} = \dfrac{1}{20}(-x_1^{(m)} \qquad\qquad + x_3^{(m-1)} + 13) \\ x_3^{(m)} = \dfrac{1}{20}(2x_1^{(m)} \quad + x_2^{(m)} \qquad\qquad + 14) \end{cases}$$

m	0	1	2	3	4
$x_1^{(m)}$	0				
$x_2^{(m)}$	0				
$x_3^{(m)}$	0				

b. Ta co

$$D - L = \begin{pmatrix} 20 & 0 & 0 \\ 1 & 20 & 0 \\ -2 & -1 & 20 \end{pmatrix} \Rightarrow (D - L)^{-1} = \begin{pmatrix} 0.05 & 0 & 0 \\ -0.0025 & 0.05 & 0 \\ 0.004875 & 0.0025 & 0.05 \end{pmatrix}$$

$$U = \begin{pmatrix} 0 & 1 & -2 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \Rightarrow T = (D - L)^{-1}U = \begin{pmatrix} 0 & 0.05 & -0.1 \\ 0 & -0.0025 & 0.055 \\ 0 & 0.004875 & -0.00725 \end{pmatrix}$$

$$c = (D - L)^{-1}b = \begin{pmatrix} 0.6 \\ 0.62 \\ 0.791 \end{pmatrix}$$

c. Coing thöic sai soi

$$\| x^{(4)} - x \| \le \frac{\| T \|}{1 - \| T \|} \| x^{(4)} - x^{(3)} \|$$

Ta coi $\|T\|_{\infty}=0.15$, nein

$$\| x^{(4)} - x \| \le \frac{0.15}{0.85} 3.5123 * 10^{-5} = 0.6199 * 10^{-5}$$

VII. Heä pt oån ñònh vaø soá ñieàu kieän :

1. Heä pt oån ñònh :

Xeùt heä phöông trình Ax = b

Ñònh nghóa :

Heä phöông trình goïi laø oån ñònh neáu moïi thay ñoåi nhoû cuûa A hay b thì nghieäm cuûa heä chæ thay ñoåi nhoû

Ví dui : Xet he phöông trình Ax = b vôi

$$A = \begin{pmatrix} 1 & 2 \\ 1 & 2.01 \end{pmatrix} \quad b = \begin{pmatrix} 3 \\ 3.01 \end{pmatrix}$$

He phöông trình co nghiem x = $(1, 1)^T$

Thay ñoi

$$b = \begin{pmatrix} 3 \\ 3.1 \end{pmatrix}$$

Nghiem cua he : x=$(-17, 10)^T$

Ta thay nghiem cua he khac rat xa khi b thay ñoi nho. Vay he khong on ñònh

Ví du : Xet he phöông trình Ax = b vôi

$$A = \begin{pmatrix} 10 & 7 & 8 & 7 \\ 7 & 5 & 6 & 5 \\ 8 & 6 & 10 & 9 \\ 7 & 5 & 9 & 10 \end{pmatrix} \quad b = \begin{pmatrix} 32 \\ 23 \\ 33 \\ 31 \end{pmatrix}$$

He co nghiem $x = (1, 1, 1, 1)^T$

Thay ñoi A mot ít

$$A = \begin{pmatrix} 10 & 7 & 8.1 & 7.2 \\ 7.08 & 5.04 & 6 & 5 \\ 8 & 5.98 & 9.98 & 9 \\ 6.99 & 4.99 & 9 & 9.98 \end{pmatrix}$$

Nghiem cua he : $x=(-81, 137, -34, 22)^T$

Ta thay nghiem cua he khac rat xa khi A thay ñoi nho. Vay he khong on ñònh

2. So ñieu kien :

Ta tìm ñieu kien ñe he on ñònh

Ñònh nghóa : So

$$\boxed{k(A) = \|A\| \, \|A\|^{-1}}$$

Goi la so ñieu kien cua ma tran A

Ta co cac tính chat :

$$(i) \quad 1 \le k(A)$$

$$(ii) \quad \frac{\|\Delta x\|}{\|x\|} \le k(A) \frac{\|\Delta b\|}{\|b\|}$$

$$(iii) \quad \frac{\|\Delta x\|}{\|x + \Delta x\|} \le k(A) \frac{\|\Delta A\|}{\|A + \Delta A\|}$$

Nhan xet :

So ñieu kien cua ma tran ñac tröng cho tính on ñònh cua he phöông trình

- k(A) cang gan 1 thì he cang on ñònh
- k(A) cang xa 1 thì he cang khong on ñònh

Ví du :

$$A = \begin{pmatrix} 1 & 2 \\ 1 & 2.01 \end{pmatrix}$$

Ta co

$$A^{-1} = \begin{pmatrix} 201 & -200 \\ -100 & 100 \end{pmatrix}$$

$\Rightarrow$ k(A) = 3.01 x 401 = 1207.01 >> 1

Vay he khong on ñònh

Ví dụ :

$$A = \begin{pmatrix} 3 & 1 & -1 \\ 1 & 2 & 1 \\ -1 & 1 & 4 \end{pmatrix}$$

Ta có

$$A^{-1} = \begin{pmatrix} 7/13 & -5/13 & 3/13 \\ -5/13 & 11/13 & -4/13 \\ 3/13 & -4/13 & 5/13 \end{pmatrix}$$

$\Rightarrow$ k(A) = 6 x 20/13 = 9.2308 >> 1

Vậy hệ không ổn định