BO MON TOAN ÖNG DUNG - ÑHBK

PHÖÔNG PHAP TÍNH – SV

CHÖÔNG 5

GIAI PHÖÔNG TRÌNH VI PHAN THÖÔNG

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NOI DUNG

A- BAI TOAN COSI (GIA TRÒ ÑAU)

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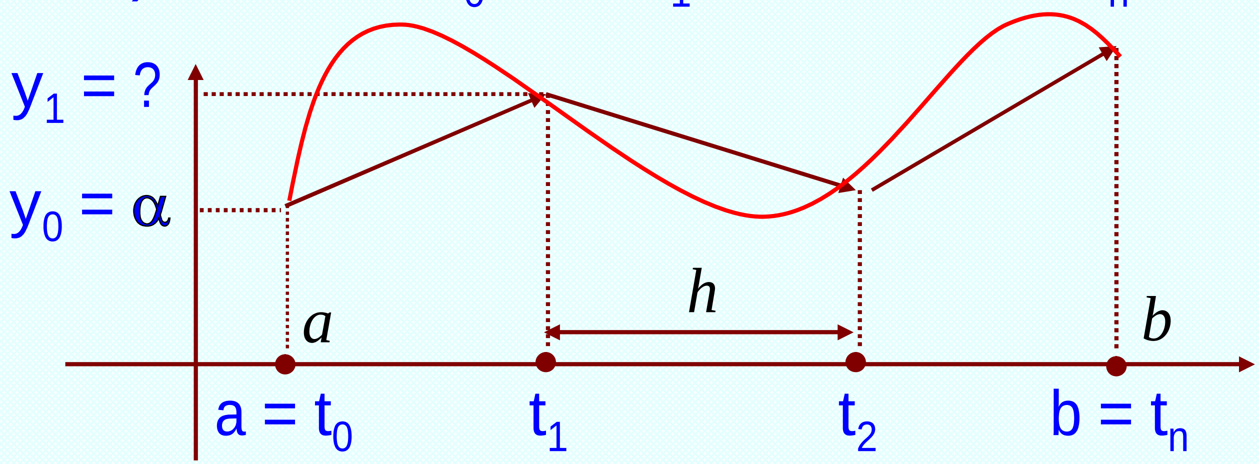
1- PHÖÔNG PHAP SAI PHAN HÖU HAN

BAI TOAN COSI

Tìm ham y = y(t) thoa phöông trình vi phan thöông & ñieu kien ñau

$$\begin{cases} y' = f(t, y), & a \le t \le b \\ y(a) = \alpha \end{cases}$$

Giai xap xæ: Chia [a, b] thanh n ñoan bang nhau, ño dai h = (b – a)/n, (n + 1) ñiem chia $t_0 = a < t_1 = a + h < \dots < t_n = b$



Can tính gan ñung gia trò $w_k \approx y_k = y(t_k)$, $k = 1 \rightarrow n$

MINH HOA Y TÖÔNG

Bai toan Cosi: $\begin{cases} y' = -5y + 5t^2 + 2t, \quad 0 \le t \\ y(0) = 1/3 \end{cases}$ Vôi böôc chia h = 0.5

& cong thöc xap xæ ñao ham 2 ñiem: $f'(x_0) \approx \dfrac{f(x_0+h) - f(x_0)}{h}$

hay tính xap xæ nghiem y tai t = 0.5, t = 1.

Tö ño xay döng ña thöc noi suy Lagrange (spline) $y_{gñ}$ va ve ño thò so sanh vôi nghiem chính xac g(t) = $t^2 + \dfrac{1}{3}e^{-5t}$

Ñiem chia:

$t_0 = 0 \qquad t_1 = 0.5 \qquad t_2 = 1.$

Ket qua tìm ñöôc: $y(0.5) = -0.5 \qquad y_{gñ.Lagrange} = at^2 + bt + c$

$y(1.0) = 1.875 \qquad \Rightarrow y_{gñ} = 6.42t^2 - 4.87t + 0.33$

CAC SÔ ÑOI GIAI XAP XE PTRÌNH VPHAN THÖÔNG

Btoan Cosi: Tìm y(t)

$$\begin{cases} y' = f(t,y), t \in [a,b] \\ y(a) = \alpha \end{cases}$$

Chia [a, b] $\to$ n ñoan

$$h = \frac{b-a}{n}, t_i = a + ih$$

Tính w_i, i = 0 $\to$ n

Sô ño Euler (i = 0 $\to$ n – 1)

$w_0 = \alpha$. Gia sö w_i ña biet $\Rightarrow$

$$w_{i+1} = w_i + hf(t_i, w_i)$$

S/ñ Euler cai tien (i = 0 $\to$ n – 1)

$w_0 = \alpha$. Gia sö w_i ña biet $\Rightarrow$

$$k_1 = hf(t_i, w_i), k_2 = hf(t_i + h, w_i + k_1)$$

$$w_{i+1} = w_i + (k_1 + k_2)/2$$

Sô ño Runge – Kutta: w_0 = α. Gia sö biet w_i $\Rightarrow$

$$\begin{cases} k_1 = hf(t_i, w_i), k_2 = hf(t_i + h/2, w_i + k_1/2) \\ k_3 = hf(t_i + h/2, w_i + k_2/2), k_4 = hf(t_{i+1}, w_i + k_3) \\ w_{i+1} = w_i + (k_1 + 2k_2 + 2k_3 + k_4)/6 \end{cases}$$

VÍ DU PHÖÔNG PHAP EULER

Bang p/phap Euler, giai bai toan Cosi vôi n = 3 ñoan chia:

$$\begin{cases} y' = y - t^2 + 1, \quad 0 \le t \le 1 \\ y(0) = 0.5 \end{cases}$$

So sanh nghiem xap xæ vôi nghiem $g(t) = (t+1)^2 - 0.5e^t$. Tö ñoi tính xap xæ tích phan bang c/t hình thang: $I = \int_0^1 y(t)dt$

Giai: $f(t,y) = y - t^2 + 1$

$h = (b-a)/n = 1/3$

$t_0 = 0,\ w_0 = 0.5 \qquad t_1,\ w_1 \qquad t_2,\ w_2 \qquad t_3,\ w_3$

Sô ño Euler:

$$\begin{cases} w_0 = 0.5 \\ w_{i+1} = w_i + hf(t_i, w_i) = w_i + 0.2(w_i - t_i^2 + 1) \end{cases}$$

KET QUA PHÖÔNG PHAP EULER

Bang ket qua:

i	t_i	w_i	$g_i = g(t_i)$	$\lvert g_i - w_i \rvert$
0	0	0.5	0.5	0
1	1/3			
2	2/3			
3	1.			

Tính gan ñung tích phan vôi cong thöc hình thang

$$\int_0^1 y(t)dt \approx \frac{h}{2}\left[y(t_0)+2y(t_1)+2y(t_2)+y(t_3)\right] \approx \frac{h}{2}\left[w_0+2w_1+2w_2+w_3\right]$$

$$=1.3528807$$

VÍ DU EULER CAI TIEN

Tính y(1.) cua bt Cosi sau bang SÑ Euler cai tien vôi h = 0.5:

$$\begin{cases} y' = y - t^2 + 1, \quad 0 \le t \le 1 \\ y(0) = 0.5 \end{cases}$$

$$f(t,y) = y - t^2 + 1\,,\ h = 0.5$$

$t_0 = 0,\ \alpha = 0.5 \qquad t_1 = 0.5,\ w_1 = ? \qquad t_2,\ w_2$

$$k_1 = hf(t_i, w_i)\,,\ k_2 = hf(t_i + h, w_i + k_1) \rightarrow w_{i+1} = w_i + \frac{k_1 + k_2}{2}$$

i	t_i	w_i	k_1	k_2
0	0.0	0.5	0.75	1.0
1	0.5	1.375	1.0625	1.21875
2	1.0	2.515625		

VÍ DU RUNGE – KUTTA

Tính y(1.) bang Runge – Kutta vôi h = 0.5

$$\begin{cases} y' = y - t^2 + 1 \\ y(0) = 0.5 \end{cases}$$

Runge – Kutta 4: w_i $\rightarrow$ w_{i+1}

$$\begin{cases} k_1 = hf(t_i, w_i),\ k_2 = hf\left(t_i + \dfrac{h}{2}, w_i + \dfrac{k_1}{2}\right) \\ k_3 = hf(t_i + h/2, w_i + k_2/2),\ k_4 = hf(t_i + h, w_i + k_3) \\ w_{i+1} = w_i + \dfrac{k_1 + 2k_2 + 2k_3 + k_4}{6} \end{cases}$$

i	t_i	w_i	k_1	k_2	k_3	k_4
0	0.0	0.5	0.75	0.90625	0.9451325	1.0976563
1	0.5	1.4251302	1.0875651	1.2032064	1.2331167	1.3286235
2	1.0	2.6396027				

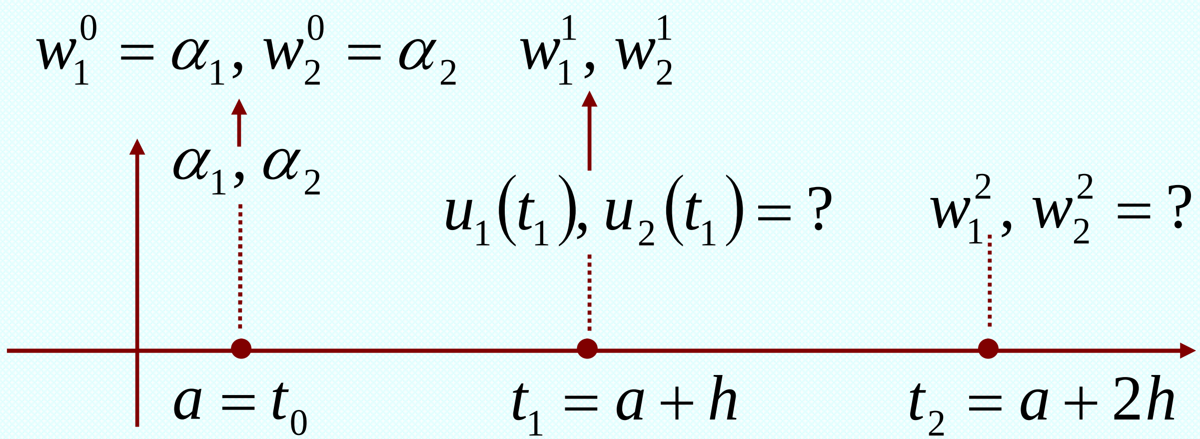
HE PHÖÔNG TRÌNH VI PHAN THÖÔNG

Bai toan Cosi : Tìm hai ham $u_1 = u_1(t)$, $u_2 = u_2(t)$ thoa

$$\begin{cases} \dfrac{du_1}{dt} = f_1(t, u_1, u_2), \quad a \le t \le b \\ \dfrac{du_2}{dt} = f_2(t, u_1, u_2), \quad a \le t \le b \end{cases}$$

& Ñieu kien ñau $\begin{cases} u_1(a) = \alpha_1 \\ u_2(a) = \alpha_2 \end{cases}$

Chia [a, b] thanh ñoan bang nhau: Phan hoach & rôi rac hoa



Ky hieu: $w_1^i \approx u_1(t_i), w_2^i \approx u_2(t_i), i \ge 0 \Rightarrow$ Biet w_1^0, w_2^0 tính w_1^i, w_2^i ?

MINH HOAÏ YÙ TÖÔÛNG

Xeùt baøi toaùn Coâsi vôùi heä phöông trình vi phaân thöôøng:

$$\begin{cases} u_1' = 3u_1 + 2u_2 - \left(2t^2+1\right)e^{2t}, \; u_1(0)=1 \\ u_2' = 4u_1 + u_2 + \left(t^2+2t-4\right)e^{2t}, \; u_2(0)=1 \end{cases}$$

Vôùi böôùc chia h = 0.5, tính xaáp xæ nghieäm u_1, u_2 taïi t = 0.5; 1

So saùnh giaù trò tính ñöôïc vôùi giaù trò nghieäm chính xaùc:

$$u_1(t) = \frac{1}{3}e^{5t} - \frac{1}{3}e^{-t} + e^{2t}; \; u_2(t) = \frac{1}{3}e^{5t} + \frac{2}{3}e^{-t} + t^2e^{2t}$$

Ñieåm chia:

$t_0 = 0$ $\quad$ $t_1 = 0.5$ $\quad$ $t_2 = 1.$

$u_1(0) = 1$ $\quad$ $u_1(0.5) = ?$ $\quad$ $u_1(1) = ?$

$u_2(0) = 1$ $\quad$ $u_2(0.5) = ?$ $\quad$ $u_2(1) = ?$

t	u_1	u_2
0	1	1
0.5		
1.0		

SÔ ÑO EULER

Bai toan Cosi : Tìm hai ham $u_1 = u_1(t)$, $u_2 = u_2(t)$ thoa

$$\begin{cases} \dfrac{du_1}{dt} = f_1(t, u_1, u_2), & a \le t \le b \\ \dfrac{du_2}{dt} = f_2(t, u_1, u_2), & a \le t \le b \end{cases}$$

& Ñieu kien ñau $\begin{cases} u_1(a) = \alpha_1 \\ u_2(a) = \alpha_2 \end{cases}$

S/ño Euler:

$w_1^0 = \alpha_1,\ w_2^0 = \alpha_2$ Gia sö biet $w_1^i, w_2^i \ (i = 0 \to n-1)$

$$w_1^{i+1} = w_1^i + hf_1\left(t_i, w_1^i, w_2^i\right), \qquad w_2^{i+1} = w_2^i + hf_2\left(t_i, w_1^i, w_2^i\right)$$

VD:

$$\begin{cases} u_1' = \underbrace{3u_1 + 2u_2 - \left(2t^2 + 1\right)e^{2t}}_{f_1(t, u_1, u_2)}, u_1(0) = 1 \\ u_2' = \underbrace{4u_1 + \ u_2 + \left(t^2 + 2t - 4\right)e^{2t}}_{f_2(t, u_1, u_2)}, u_2(0) = 1 \end{cases}$$

$$\Rightarrow \begin{cases} \alpha_1 = 1 \\ \alpha_2 = 1 \end{cases} \Rightarrow \begin{cases} w_1^0 = 1 \\ w_2^0 = 1 \end{cases}$$

$$w_1^1 = 1 + 0.5 f_1(0,1,1) =$$

$$w_2^1 = 1 + 0.5 f_2(0,1,1) =$$

AP DUNG : PHÖÔNG TRÌNH VI PHAN CAP 2

Bai toan Cosi cap 2 (Ph/trình vi phan cap 2 va ñkien ñau):

$$\begin{cases} y'' = f(t, y, y'), \quad t \geq a \\ y(a) = \alpha_1, \; y'(a) = \alpha_2 \end{cases}$$

Ñöa ve bai toan Cosi cap 1: Ñoi bien $u_1(t) = y(t)$, $u_2(t) = y'(t)$

$$\begin{cases} u_1' = u_2 = f_1(t, u_1, u_2) \\ u_2' = y'' = f(t, y, y') = f_2(t, u_1, u_2) \end{cases}$$

Ñieu kien ñau: $\begin{cases} u_1(a) = y(a) = \alpha_1 \\ u_2(a) = y'(a) = \alpha_2 \end{cases} \Rightarrow \begin{cases} w_1^0 = \alpha_1 \\ w_2^0 = \alpha_2 \end{cases}$

Sô ño Euler: $\begin{cases} w_1^1 = w_1^0 + hf_1\left(t_0, w_1^0, w_2^0\right) = \alpha_1 + h\alpha_2 \\ w_2^1 = w_2^0 + hf_2\left(t_0, w_1^0, w_2^0\right) = \alpha_2 + hf_2(a, \alpha_1, \alpha_2) \end{cases}$

VÍ DU

Vôi h = 0.1, tính xap xæ gia trò y(0.2), y'(0.2) cua nghiem bai toan sau bang phöông phap Euler:

$$\begin{cases} y''-2y'+2y = e^{2t}\sin t \,,\ t \geq 0 \\ y(0) = -0.4, \quad y'(0) = -0.6 \end{cases}$$

Ñoi bien ñöa ve bai toan Cosi cap 1: $u_1 = y(t)$, $u_2 = y'(t) \Rightarrow$

$$\begin{cases} u_1' = u_2 = f_1(t, u_1, u_2) \quad \& \quad u_1(0) = -0.4 \,,\, u_2(0) = -0.6 \\ u_2' = -2y + 2y' + e^{2t}\sin t = -2u_1 + 2u_2 + e^{2t}\sin t = f_2(t, u_1, u_2) \end{cases}$$

$$\begin{cases} w_1^0 = -0.4 \\ w_2^0 = -0.6 \end{cases} \Rightarrow \begin{cases} w_1^1 = w_1^0 + hf_1\left(t_0, w_1^0, w_2^0\right) = -0.4 + 0.1 f_1(0,-0.4,-0.6) \\ w_2^1 = w_2^0 + hf_2\left(t_0, w_1^0, w_2^0\right) = -0.6 + 0.1 f_2(0,-0.4,-0.6) \end{cases}$$

BAØI TOAÙN BIEÂN

Baøi toaùn bieân caáp 2: Tìm haøm y = y(x) thoaû phöông trình

$$\begin{cases} y'' = f(x, y, y'), \quad a \le x \le b \\ y(a) = \alpha, \ \ y(b) = \beta \end{cases}$$

Hay gaëp: Baøi toaùn bieân tuyeán tính caáp 2

$$\begin{cases} y'' = p(x)y' + q(x)y + r(x), \quad a \le x \le b \\ y(a) = \alpha, \ y(b) = \beta \end{cases}$$

MINH HOA

Tính gia trò nghiem y cua bai toan bien tuyen tính cap 2

$$\begin{cases} y'' = -(x+1)y' + 2y + (1-x^2)e^{-x}, \; 0 \le x \le 1 \\ y(0) = -1, \; y(1) = 0 \end{cases}$$

tai cac ñiem chia cach ñeu cua [0, 1] vôi böôc chia h = 1/3

va xap xæ ñao ham y', y'' bang cong thöc höông tam

Ñiem chia:

$x_0 = 0$ $\quad x_1 = 1/3$ $\quad x_2 = 2/3$ $\quad x_3 = 1$

$$y(0) = y_0 = -1 \qquad y\left(\frac{1}{3}\right) = y_1 = ? \qquad y\left(\frac{2}{3}\right) = y_2 = ? \qquad y(1) = y_3 = 0$$

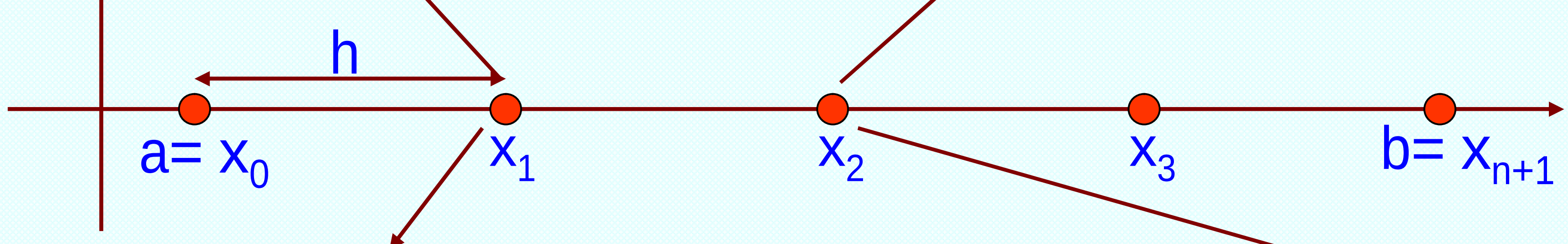
PHÖÔNG PHAP SAI PHAN HÖU HAN

BT bien tuyen tính
$$\begin{cases} y'' = p(x)y' + q(x)y + r(x),\ a \le x \le b \ (*) \\ y(a) = \alpha,\ y(b) = \beta \end{cases}$$

Chia [a, b] thanh cac ñoan nhoi bang nhau. Thay x = x_k vao (*).

Xap xæ $y'(x_k)$, $y''(x_k)$: cong thöc ñao ham höông tam

$$y'(x_1) \approx \frac{y(x_2) - y(x_0)}{2h} \qquad y'(x_2) \approx \frac{y(x_3) - y(x_1)}{2h}$$



$$y''(x_1) \approx \frac{y(x_2) - 2y(x_1) + y(x_0)}{h^2} \qquad y''(x_2) \approx \frac{y(x_3) - 2y(x_2) + y(x_1)}{h^2}$$

CONG THÖC LAP GHEP

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n moc $x_k \in (a, b)$ – öng n gia trò y_k chöa biet $\rightarrow$ Ma tran cap n

Ky hieu $p_k = p(x_k) \ldots y_k = y(x_k)$, $1 \le k \le n \Rightarrow y = [y_1, \ldots y_n]^T$: $Ay = b$

$$A = \begin{bmatrix} 2+h^2q_1 & -1+\frac{h}{2}p_1 & 0 & \ldots\ldots & 0 \\ -1-\frac{h}{2}p_2 & 2+h^2q_2 & -1+\frac{h}{2}p_2 & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & -1+\frac{h}{2}p_{n-1} \\ 0 & \ldots\ldots & 0 & -1-\frac{h}{2}p_n & 2+h^2q_n \end{bmatrix} \quad b = \begin{bmatrix} -h^2r_1 + \left(1+\frac{h}{2}p_1\right)\alpha \\ -h^2r_2 \\ -h^2r_3 \\ \ldots\ldots\ldots\ldots \\ -h^2r_{n-1} \\ -h^2r_n + \left(1-\frac{h}{2}p_n\right)\beta \end{bmatrix}$$

LAÄP BAÛNG LAËP GHEÙP

BT bieân tuyeán tính $\begin{cases} y'' = p(x)y' + q(x)y + r(x),\ a \le x \le b\ (*) \\ y(a) = \alpha,\ y(b) = \beta \end{cases}$

❖ Chia [a, b] thaønh caùc ñoaïn nhoû ñoä daøi h. n ñieåm chia x_k (khoâng keå 2 ñaàu) – öùng vôùi y_k chöa bieát $\rightarrow$ n aån soá y_k

❖ Laäp baûng coät $x_k \rightarrow p_k = p(x_k)$, $q_k = q(x_k)$, $r_k = r(x_k) \rightarrow a_{kk}$ (ñ/cheùo chính), $a_{k,k+1}$ (cheùo treân), $a_{k-1,k}$ (döôùi), $b_k \rightarrow$ Nghieäm y_k

❖ Ñ/cheùo a_{kk}: $k = 1 \rightarrow n$; $a_{k,k+1}$: $k = 1 \rightarrow (n-1)$, $a_{k-1,k}$: $k = 2 \rightarrow n$

i	x_k	p_k	q_k	r_k	a_{kk}	$a_{k,k+1}$	$a_{k-1,k}$	b_k	y_k
1									
2									
3									

VÍ DU

Giai bai toan bien cap 2 sau bang phöông phap sai phan höu han vôi böôc chia h = 0.2

$$\begin{cases} y'' = -3y' + 2y + 2x + 3 \\ y(0) = 2, \quad y(1) = 1 \end{cases}$$

h = 0.2 $\Rightarrow$ n = 5 $\Rightarrow$ 6 ñiem chia $\Rightarrow$ He phöông trình 4 an

Ma tran cap 4: Cheo chính a_{kk} – 4 phan tö; Cheo tren $a_{k,\,k+1}$: 3

i	x_i	p_i	q_i	r_i	a_{ii}	$a_{i,i+1}$	$a_{i-1,i}$	b_i
1	0.2	-3	2	3.4	-2.08	1.3		-1.264
2	0.4	-3	2	3.8	-2.08	1.3	0.7	0.152
3	0.6	-3	2	4.2	-2.08	1.3	0.7	0.168
4	0.8	-3	2	4.6	-2.08		0.7	-1.116

KET QUA

Giai he bang phep khö Gauss (lam tron 3 chö so le):

$$[A|b] = \left[\begin{array}{cccc|c} 2.08 & -1.3 & 0 & 0 & 1.264 \\ -0.7 & 2.08 & -1.3 & 0 & -0.152 \\ 0 & -0.7 & 2.08 & -1.3 & -0.168 \\ 0 & 0 & -0.7 & 2.08 & 1.116 \end{array}\right]$$

$$\rightarrow [A|b] = \left[\begin{array}{cccc|c} 1 & -0.625 & 0 & 0 & 0.608 \\ 0 & 1.642 & -1.3 & 0 & 0.273 \\ 0 & -0.7 & 2.08 & -1.3 & -0.168 \\ 0 & 0 & -0.7 & 2.08 & 1.116 \end{array}\right] \quad \Rightarrow y = \begin{bmatrix} 1.006 \\ 0.636 \\ 0.593 \\ 0.736 \end{bmatrix}$$