BO MON TOAN ÖNG DUNG - ÑHBK

PHÖÔNG PHAP TÍNH – HK 2 0506

CHÖÔNG 3

NOI SUY VA BÌNH PHÖÔNG CÖC TIEU

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NOÄI DUNG

1- NOÄI SUY ÑA THÖÙC LAGRANGE

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BAI TOAN TONG QUAT VE NOI SUY

<u>Noi suy</u>: Bang chöa (n+1) cap dö lieu $\{ (x_k, y_k) \}$, $k = 0 \rightarrow n$

Moc noi suy	x_0	x_1	…	$x = \alpha \notin \{ x_k \}$	…	x_{n-1}	x_n
Gia trò noi suy	y_0	y_1	…	$y = ?$	…	y_{n-1}	y_n

x_k : moc noi suy, y_k : gia trò (ham) noi suy

Tö bang nay, noi suy gia trò y_{bang} tai ñiem $x = \alpha$?

Noi suy ña thöc: Xac ñònh ña thöc $y = P(x)$ thoa ñieu kien noi suy $P(x_k) = y_k$, $k = 0 \ldots n \Rightarrow y_{bang} \approx P(\alpha)$

NOI SUY ÑA THÖC LAGRANGE

Bang chöa (n+1) cap so lieu $\{(x_k,y_k)\}$, $k = 0 \to n$

$\exists!$ ña thöc $L(x)$, bac $\le n$, thoa ñ/kien noi suy $L(x_k) = y_k$, $k = 0 \dots n$

<u>Tìm ña thöc noi suy</u> Minh hoa bang 3 dö lieu: $\{(x_k,y_k)\}$, $k=0\to2$

Moc noi suy x_k	2	2.5	4
Gia Trò noi suy y_k	0.5	0.4	0.25

Tai $x = 3$, $y_{bang} \approx ?$

Cach 1: 3 moc $\Rightarrow n = 2 \Rightarrow L(x) = ax^2 + bx + c$ (3 he so can tìm)

$$\begin{cases} L(2) = 0.5 \\ L(2.5) = 0.4 \\ L(4) = 0.25 \end{cases} \Leftrightarrow \begin{cases} 4a + 2b + c = 0.5 \\ 6.25a + 2.5b + c = 0.4 \\ 16a + 4b + c = 0.25 \end{cases}$$

$y_{bang} \approx L(3) = 0.325$

VÍ DU SAI SO

Sai so: $\left|f(x)-L(x)\right| \le \dfrac{\max\limits_{[a,b]}\left|f^{(n+1)}(x)\right|}{(n+1)!}\left|(x-x_0)\dots(x-x_n)\right| = \Delta_{\text{Noi suy tai } x}$

Öôc löông sai so cua viec xap xæ gia trò $\sqrt{115}$ bang ña thöc noi suy Lagrange bac hai ham y = $\sqrt{x}$ xay döng tai cac moc x_0 = 100, x_1 = 121, x_2 = 144. Yeu cau: Lam tron ket qua (sai so) ñen chö so le thö 4

Giai: $f(x)=\sqrt{x} \Rightarrow M = \max\limits_{[a,b]}\left|f^{(3)}(x)\right| = \max\limits_{[100,144]} \dfrac{3}{8}x^{-5/2}$

$$\left|f(115)-L(115)\right| \le \mathrm{M}\cdot\frac{1}{3!}\cdot\left|(115-100)(115-121)(115-144)\right|$$

Ket qua: Nhac lai: Sai so: luon lam tron len!

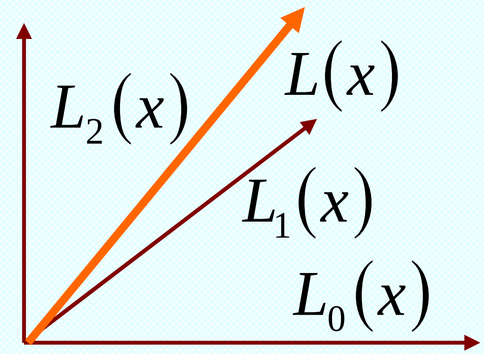
NHIEU MOC → ÑA THÖC NOI SUY CÔ SÔ

Ña thöc noi suy cô sô tai x_k: $L_k(x_k) = 1$, $L_k(x_i) = 0\ \forall\ i \neq k$

Moc NS	2	2.5	4
Gia Trò NS	0.5	0.4	0.25
ÑTNSCS $L_0(x)$	1	0	0
ÑTNSCS $L_1(x)$	0	1	0
ÑTNSCS $L_2(x)$	0	0	1

3 moc ⇒ 3 ÑT NSCS

$$L_0(x) = \frac{(x-2.5)(x-4)}{(2-2.5)(2-4)}$$



$$L_1(x) = \frac{(x-2)(x-4)}{(2.5-2)(2.5-4)} \qquad L_2(x) = \frac{(x-2)(x-2.5)}{(4-2)(4-2.5)}$$

Ña thöc noi suy: $L(x) = 0.5L_0(x) + 0.4L_1(x) + 0.25L_2(x)$

Thiet lap cong thöc tong quat vôi (n + 1) moc $\{(x_k, y_k)\}$?

CONG THÖC TONG QUAT

(n+1) moc $\Rightarrow$ (n+1) ña thöc noi suy cô sôi. Ña thöc noi suy cô sôi $L_k(x)$ tai x_k ($k = 0 \ldots n$): $L_k(x_k) = 1$, $L_k(x_i) = 0\ \forall\, i \neq k$:

$$\begin{cases} L_k(x_k) = 1 \\ L_k(x_0) = L_k(x_1) = \ldots = L_k(x_{k-1}) = L_k(x_{k+1}) = \ldots = L_k(x_n) = 0 \end{cases}$$

$$L_k(x) = \frac{(x - x_0)\ldots(x - x_{k-1})(x - x_{k+1})\ldots(x - x_n)}{(x_k - x_0)\ldots(x_k - x_{k-1})(x_k - x_{k+1})\ldots(x_k - x_n)},\ 0 \le k \le n$$

$\Rightarrow$ Ña thöc noi suy $L(x) = y_0 L_0(x) + y_1 L_1(x) + \cdots + y_n L_n(x)$

Öu ñiem: Cong thöc tong quat cho ña thöc noi suy L(x)

Chæ phu thuoc bo moc $\{x_k\}$ ($0 \le k \le n$), khong phu thuoc y_k

VÍ DU

Bang 4 moc 1, 2, 3, 4 ; 4 gia trò 5, 7, 8, 9. Viet ra bieu thöc cac ña thöc noi suy cô sô. Tính gia trò bang tai x = 3.5?

$$L_0(x)=\frac{(x-2)(x-3)(x-4)}{(1-2)(1-3)(1-4)} \Rightarrow L_0(3.5)=0.0625$$

$$L_1(3.5)=\frac{(3.5-1)(3.5-3)(3.5-4)}{(2-1)(2-3)(2-4)}=$$

$$L_2(3.5)=$$

$$L_3(3.5)=$$

$$L(x)=5L_0(x)+7L_1(x)+8L_2(x)+9L_3(x) \Rightarrow L(3.5)=8.4375$$

Viet bieu thöc $L_k(x)$ (Khong tính!) Thay x $\rightarrow$ Gia trò

NOI SUY NEWTON – MOC CACH ÑEU

Bang $\{(x_k,y_k)\}$, $k = 0 \rightarrow n$, moc noi suy cach ñeu: x_0, $x_1 = x_0 + h$, $x_2 = x_1 + h \dots x_n = x_{n-1} + h$. Lap bang sai phan :

Moc NS	Gtrò NS	Δy_k	$\Delta^2 y_k$	$\Delta^3 y_k$
x_0	y_0			
		Δy_0		
x_1	y_1		$\Delta^2 y_0$	
		Δy_1		$\Delta^3 y_0$
x_2	y_2		$\Delta^2 y_1$	
		Δy_2		
x_3	y_3			

Cap 1: $\Delta y_k = y_{k+1} - y_k$

Ví dui: $\Delta y_0 = y_1 - y_0$

$\Delta^2 y_k = \Delta y_{k+1} - \Delta y_k \dots$

VD: Bang sai phan 3 moc (cach ñeu)

x_k	y_k	Δy	$\Delta^2 y$
1	2		
		2	
2	4		1
		3	
3	7		

ÑA THÖÙC NOÄI SUY NEWTON

Ña thöùc noäi suy Newton tieán: x $\approx x_0$ (ñaàu baûng)

$$t = \frac{x - x_0}{h}$$ ⇔ $x = x_0 + th$ ⇒ Ña thöùc noäi suy tieán:

$$N(x) = y_0 + t\Delta y_0 + \frac{t(t-1)}{2!}\Delta^2 y_0 + \ldots + \frac{t(t-1)\ldots(t-n+1)}{n!}\Delta^n y_0$$

Ña thöùc theo t & Sai phaân naèm treân ñöôøng cheùo tieán

Ña thöùc noäi suy Newton luøi: x $\approx x_n$ (cuoái baûng)

$$t = \frac{x - x_n}{h}$$ ⇔ $x = x_n + th$ ⇒ Ña thöùc noäi suy luøi:

$$N(x) = y_n + t\Delta y_{n-1} + \frac{t(t+1)}{2!}\Delta^2 y_{n-2} + \ldots + \frac{t\ldots(t+n-1)}{n!}\Delta^n y_0$$

Sai phaân naèm treân ñöôøng cheùo luøi (töø cuoái baûng ñi leân)

VÍ DU NOI SUY NEWTON

Cho bang gia trò sinx tö 15° → 55°. Xay döng ña thöc noi suy tien (lui) cap 3 & tính sin16° (sin54°)

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$
15	0.2588			
20	0.3420			
25	0.4226			
30	0.5			
35	0.5736			
40	0.6428			
45	0.7071			
50	0.7660			
55	0.8192			

VÍ DU NOI SUY NEWTON

Ña thöc noi suy tien: x ≈ 15° $\Rightarrow t = \dfrac{x-15}{5} \Leftrightarrow x = 15 + 5t$

$$N_1(t) = 0.2588 + 0.0832t - 0.0026\,\frac{t(t-1)}{2} - 0.0006\,\frac{t(t-1)(t-2)}{3!}$$

x = 16° $\Rightarrow$ t = 0.2 $\Rightarrow$ $N_1(0.2)$ = 0. 2756 sin16° = 0. 2756

Ña thöc noi suy lui: x ≈ 55° $\Rightarrow t = \dfrac{x-55}{5} \Leftrightarrow x = 55 + 5t$

$$N_2(t) = 0.8192 + 0.0532t - 0.0057\,\frac{t(t+1)}{2} - 0.0003\,\frac{t(t+1)(t+2)}{3!}$$

x = 54° $\Rightarrow$ t = –0.2 $\Rightarrow$ $N_2(-0.2)$ = 0.80903 sin54° = 0. 8090

Cau hoi: Tính tai x = 54° vôi Noi suy tien. Nhan xet?

Tat ca sai phan: Noi suy Newton ≡ Lagrange!

HIEN TÖÔNG RUNGE

Noi suy ham $f(x) = 1/(1+25x^2)$, $x \in [-1, 1]$ bang ña thöc noi suy, 5 moc cach ñeu. Tính L(0.95), so sanh gia trò tính ñöôc vôi gia trò chính xac f(0.95)

Lap bang noi suy: 5 moc cach ñeu tren $[-1, 1] \Rightarrow$

$x_0 = -1$, $x_1 = -0.5$, $x_2 = 0$, $x_3 = 0.5$, $x_4 = 1$ & $y_k = f(x_k)$

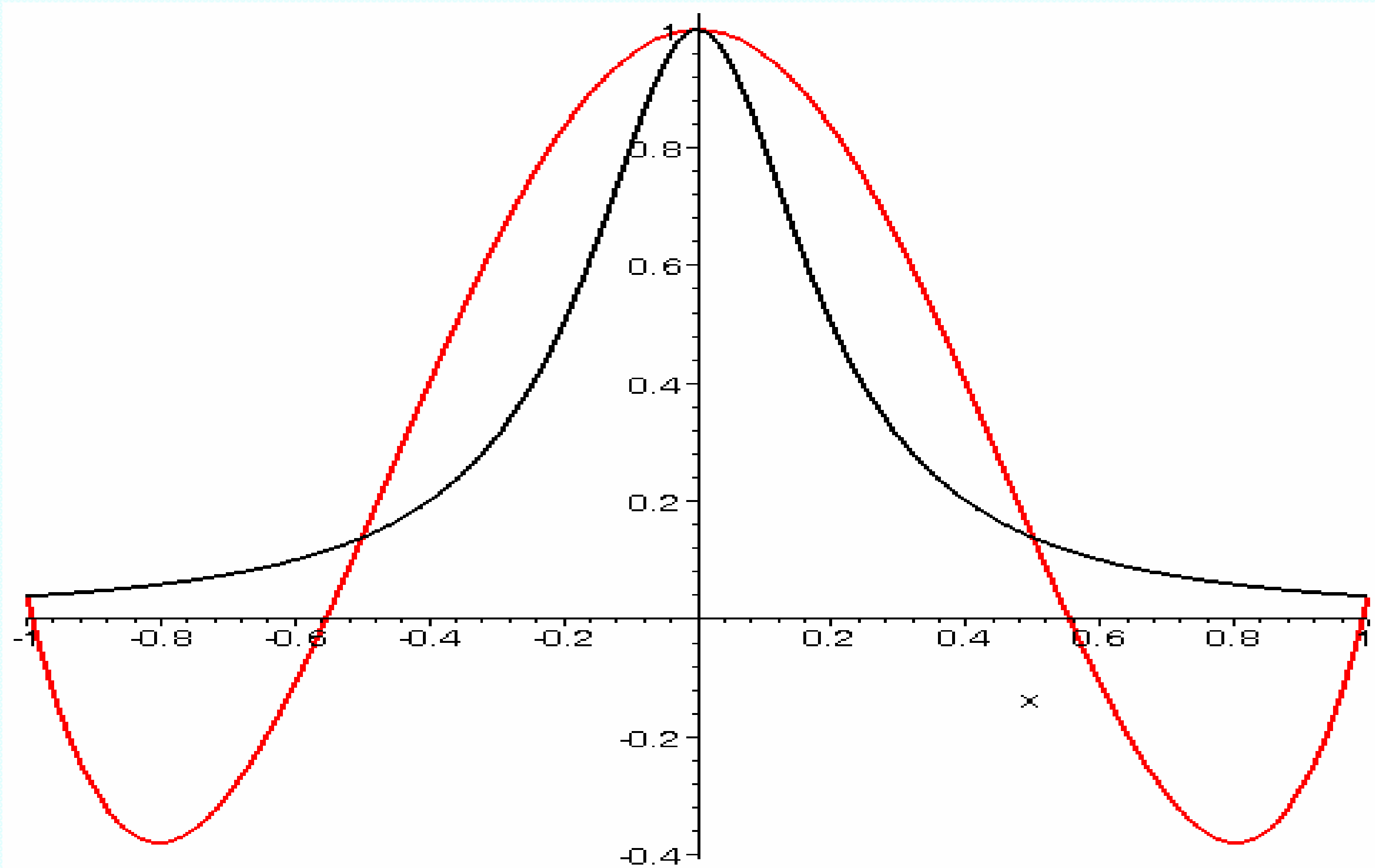
x_k	–1	–0.5	0.	0.5	1.
y_k	0.038	0.138	1	0.138	0.038

Gia trò L(0.95) =

Gia trò chính xac f(0.95) = 0.04

KET QUA

So sanh ño thò ham ban ñau f(x) va ña thöc noi suy $P_4(x)$

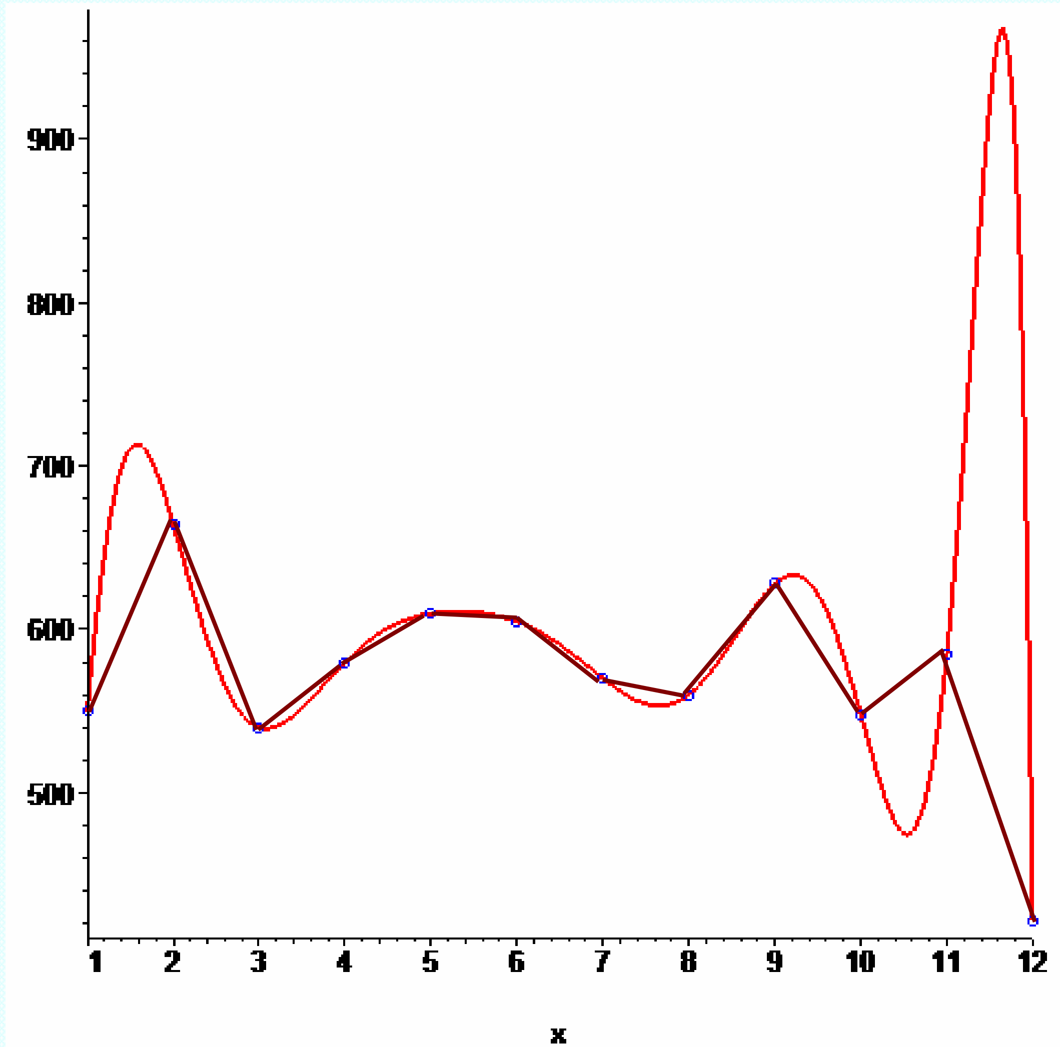


Tang so nut co the khien sai so tang!

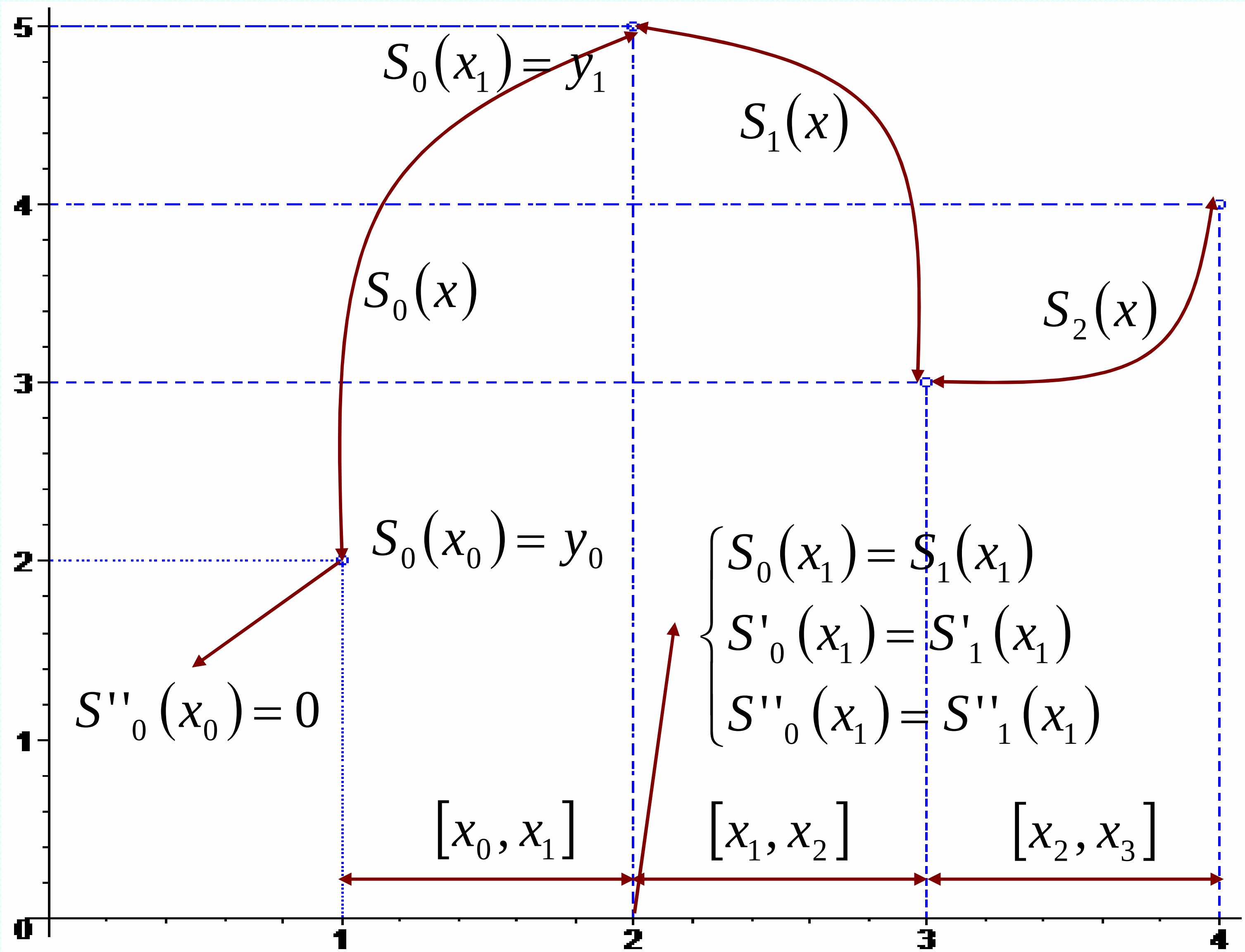
NOI SUY GHEP TRÔN

Noi suy Lagrange: Bac qua lôn ⇒ Ñoi thò phöc tap

Thay ña thöc noi suy bac n bang ña thöc noi suy bac thap (bac 1, 2, 3 …) tren töng ñoan $[x_k, x_{k+1}]$, $k = 0 \ldots n - 1$



YÙ TÖÔÛNG NOÄI SUY GHEÙP TRÔN BAÄC 3



XAÂY DÖÏNG HAØM NOÄI SUY GHEÙP TRÔN BAÄC 3

Tìm haøm baäc 3 treân töøng ñoaïn, lieân tuïc vaø coù ñaïo haøm ñeán caáp 2 noäi suy baûng soá lieäu sau:

x	1	2	3
y	2	3	–4

Haøm noäi suy:

$$S(x) = \begin{cases} S_0(x) = a_0 + b_0 x + c_0 x^2 + d_0 x^3, \ x \in [1,2] \\ S_1(x) = a_1 + b_1 x + c_1 x^2 + d_1 x^3, \ x \in [2,3] \end{cases}$$

Daïng thuaän tieän hôn:

$$S(x) = \begin{cases} S_0(x) = a_0 + b_0(x-1) + \dots, \ x \in [1,2] \\ S_1(x) = a_1 + b_1(x-2) + \dots, \ x \in [2,3] \end{cases}$$

NOI SUY SPLINE (GHEP TRÔN) BAC 3

1/ Ham dang bac 3 tren töng ñoan $[x_k, x_{k+1}]$, $k = 0 \to n-1$

$$S = \begin{cases} S_0 = a_0 + b_0(x - x_0) + c_0(x - x_0)^2 + d_0(x - x_0)^3, x \in [x_0, x_1] \\ S_1 = a_1 + b_1(x - x_1) + c_1(x - x_1)^2 + d_1(x - x_1)^3, x \in [x_1, x_2] \\ \ldots\ldots \\ S_{n-1} = a_{n-1} + b_{n-1}(x - x_{n-1}) + c_{n-1}(x - x_{n-1})^2 + \cdots, x \in [x_{n-1}, x_n] \end{cases}$$

2/ Ñieu kien noi suy: $S(x_k) = y_k$, $k = 0, 1 \ldots n$

3/ Ghep trôn:

$$\begin{cases} S_k(x_{k+1}) = S_{k+1}(x_{k+1}) \\ S_k'(x_{k+1}) = S_{k+1}'(x_{k+1}) \quad k = 0 \to (n-2) \\ S_k''(x_{k+1}) = S_{k+1}''(x_{k+1}) \end{cases}$$

4/ Ñieu kien bien töi nhien: $S''(x_0) = S''(x_n) = 0$

GIAI THUAT NOI SUY SPLINE BAC 3

I/ Ño dai $h_k = x_{k+1} - x_k$, k = 0 … n –1. He so $a_k = y_k$, k = 0 … n

II/ $c = [c_0, \ldots c_n]^T$ la nghiem ($c_n = S''(x_n)/2$) he Ac = e vôi

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 & \cdots\cdots & 0 \\ h_0 & 2(h_0+h_1) & h_1 & 0 & \cdots\cdots & 0 \\ 0 & h_1 & 2(h_1+h_2) & h_2 & 0\cdots & \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & & h_{n-2} & 2(h_{n-2}+h_{n-1}) & h_{n-1} \\ 0 & \cdots\cdots & & & 0 & 1 \end{bmatrix} \qquad e = \begin{bmatrix} 0 \\ \dfrac{3(a_2-a_1)}{h_1} - \dfrac{3(a_1-a_0)}{h_0} \\ \dfrac{3(a_3-a_2)}{h_2} - \dfrac{3(a_2-a_1)}{h_1} \\ \cdots\cdots\cdots\cdots\cdots \\ \dfrac{3(a_n-a_{n-1})}{h_{n-1}} - \dfrac{3(a_{n-1}-a_{n-2})}{h_{n-2}} \\ 0 \end{bmatrix}$$

Böôc III: $$b_k = \frac{a_{k+1}-a_k}{h_k} - \frac{h_k(2c_k + c_{k+1})}{3}, k = 0 \ \ldots \ n-1$$

$$d_k = \frac{c_{k+1}-c_k}{3h_k}, \quad k = 0 \ \cdots \ n-1$$

VÍ DUÏ NOÄI SUY SPLINE (GHEÙP TRÔN) BAÄC 3

Laäp haøm noäi suy spline baäc 3 g(x) thoaû ñieàu kieän bieân töï nhieân vaø noäi suy baûng sau

Moác NS	$x_0 = 1$	$x_1 = 2$	$x_2 = 3$	$x_3 = 4$
Giaù trò NS	$y_0 = 2$	$y_1 = 1$	$y_2 = 3$	$y_3 = 2$

Haøm spline
$$S = \begin{cases} a_0 + b_0(x-1) + c_0(x-1)^2 + d_0(x-1)^3, \ x \in [1,2] \\ a_1 + b_1(x-2) + c_1(x-2)^2 + d_1(x-2)^3, \ x \in [2,3] \\ a_2 + b_2(x-3) + c_2(x-3)^2 + d_2(x-3)^3, \ x \in [3,4] \end{cases}$$

Böôùc I: Ñoä daøi böôùc chia $h_0 = h_1 = h_2 = 1.$

Heä soá: $a_k = y_k, 0 \le k \le 3 \Rightarrow a_0 = 2, a_1 = 1, a_2 = 3, a_3 = 2$

BAÛNG TÍNH NOÄI SUY SPLINE (GHEÙP TRÔN) BAÄC 3

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k	h_k	a_k	b_k	e_k	c_k	d_k
0	1	2		0	**0**	
1	1	1				
2	1	3				
3		2		0	**0**	

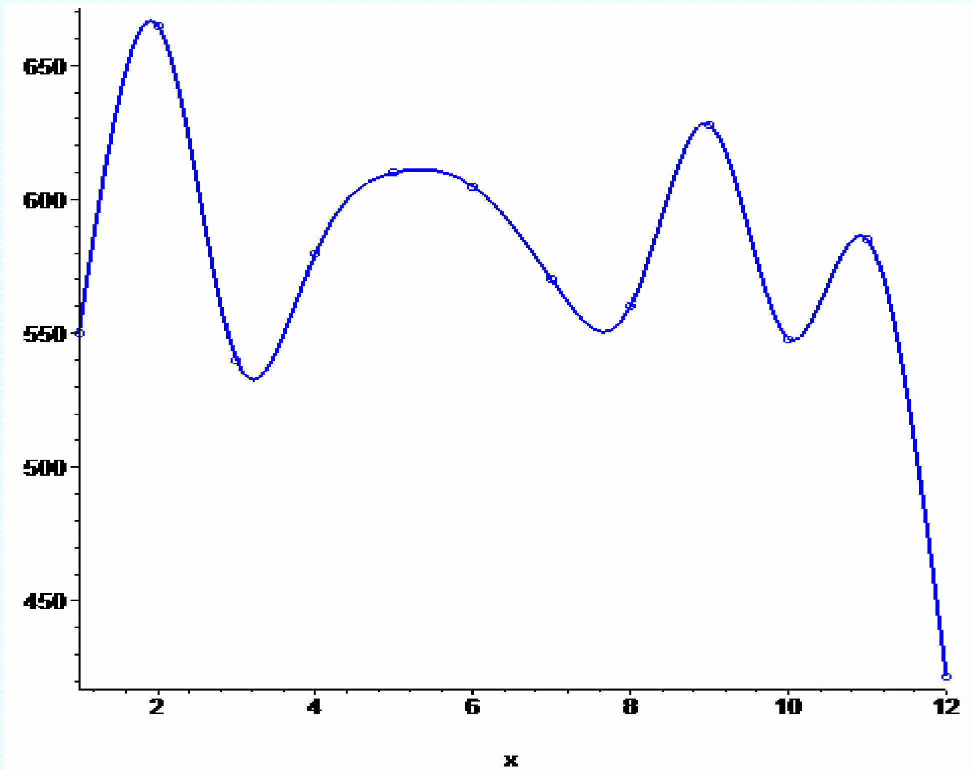
Böôùc II: $c_3 = g''(x_3)/2 \Rightarrow$ **c = [c_0, c_1, c_2, c_3]^T laø nghieäm**

III/ b_k, d_k, $0 \le k \le 2$:
$$d_0 = \frac{c_1 - c_0}{3h_0} \ldots \quad b_0 = \frac{a_1 - a_0}{h_0} - \frac{h_0(2c_0 + c_1)}{3} \ldots$$

BÌNH PHÖÔNG CÖC TIEU

Thöc nghiem: Thong ke löông möa 12 thang & ve ño thò

Thang	1	2	3	4	5	6	7	8
Löông möa	550	665	540	580	610	605	570	…

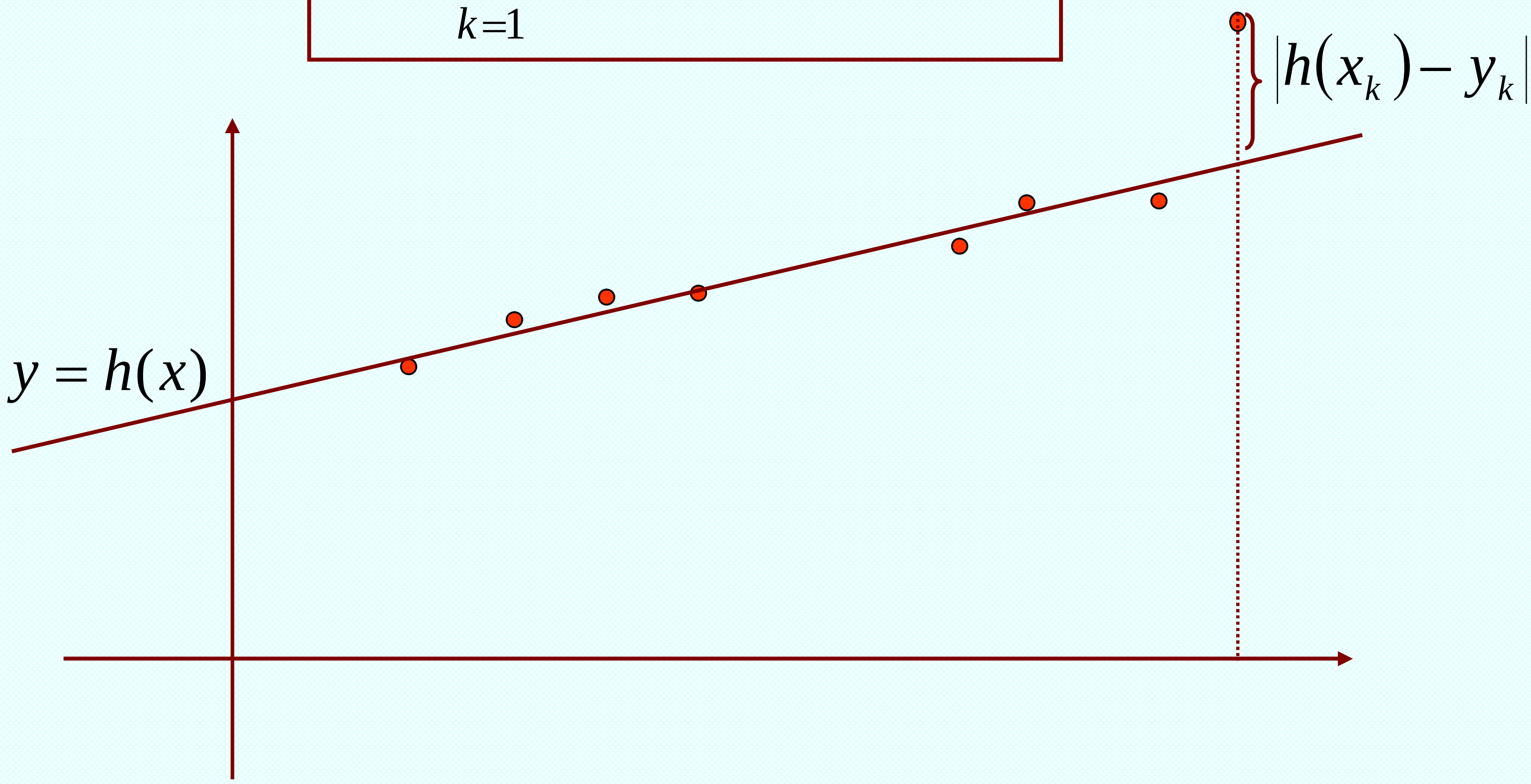


PHÖÔNG PHAP BÌNH PHÖÔNG CÖC TIEU (BPCT)

Nhieu dö lieu & y_k co sai so: Ap ñat $L(x_k) = y_k$: vo nghóa!

Giai quyet: h(x) xap xæ bang $\{(x_k, y_k)\}$ theo nghóa BPCT

$$\boxed{F = \sum_{k=1}^{n}\left[h(x_k) - y_k\right]^2 \rightarrow \min}$$



TRÖÔNG HÔP TUYEN TÍNH

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h tuyen tính: h(x) = ax + b $\Rightarrow F(a,b) = \sum_{k=1}^{n} [ax_k + b - y_k]^2$

Ñiem döng: $\begin{cases} \partial F / \partial a = 0 \\ \partial F / \partial b = 0 \end{cases} \Rightarrow \begin{cases} a \sum_{k=1}^{n} x_k^2 + b \sum_{k=1}^{n} x_k = \sum_{k=1}^{n} x_k y_k \\ a \sum_{k=1}^{n} x_k + nb = \sum_{k=1}^{n} y_k \end{cases}$

Giai he 2 phöông trình 2 an tìm a, b. So vôi ñöông cong y = $h_1(x)$ ⇔ Tong S = $\Sigma (h_1(x_k) - y_k)^2$: cang be cang tot

VD: Tìm ham bac 1 xap x bang sau theo nghóa BPCT

x_k	1	2	3	4	5	6	7	8	9	10
y_k	1.3	3.5	4.2	5.0	7.0	8.8	10.1	12.5	13.	15.6

ÑA THÖC BÌNH PHÖÔNG CÖC TIEU BAC CAO

h(x) = ax^2 + bx + c $\Rightarrow F(a,b,c) = \sum_{k=1}^{n} \left[a x_k^2 + b x_k + c - y_k \right]^2$

Ñiem döng: $\begin{cases} \partial F / \partial a = 0 \\ \partial F / \partial b = 0 \\ \partial F / \partial c = 0 \end{cases} \Rightarrow \begin{cases} a \sum_{k=1}^{n} x_k^4 + b \ldots + c \ldots = \ldots \\ a \sum_{k=1}^{n} x_k^3 + b \ldots + c \ldots = \ldots \\ a \sum_{k=1}^{n} x_k^2 + b \ldots + nc \quad = \ldots \end{cases}$

Tong quat: Ñiem döng ham tong bình phöông ñoi lech

h = ax^2 + bx $\Rightarrow F(a,b) = \sum_{k=1}^{n} \left[a x_k^2 + b x_k - y_k \right]^2 \Rightarrow \frac{\partial F}{\partial a} = \frac{\partial F}{\partial b} = 0$

HAØM MUÕ

$y = h(x) = be^{ax} \Rightarrow$ lny = ax + lnb $\Rightarrow$ Töông quan baäc 1 giöõa $\ln y_k$ & x_k. Laäp baûng $\{(x_k, \ln y_k)\}$ xaùc ñònh a & lnb.

$$F(a,b) = \sum_{k=1}^{n} [ax_k + \ln b - \ln y_k]^2 \Rightarrow \begin{cases} a\sum_{k=1}^{n} x_k^2 + \ln b \sum_{k=1}^{n} x_k = \\ a\sum_{k=1}^{n} x_k + n\ln b \quad = \end{cases}$$

VD: Xaáp xæ baèng soá vôùi p/phaùp bình phöông cöïc tieåu

k	x_k	y_k	$\ln y_k$	x_k^2	$x_k \ln y_k$
1	1.00	5.10	1.629	1.0000	1.629
2	1.25	5.79	1.756	1.5625	2.195
3	1.50	6.53	1.876	2.2500	2.814
4	1.75	7.45	2.008	3.0625	3.514
5	2.00	8.46	2.135	4.0000	4.270
Σ	7.50		9.404	11.875	14.422