BO MON TOAN ÖNG DUNG - ÑHBK

PHÖÔNG PHAP TÍNH – BG SINH VIEN

CHÖÔNG 2

HE PHÖÔNG TRÌNH TUYEN TÍNH

$$Ax = b$$

TS. NGUYEN QUOC LAN (2/2006)

NOI DUNG

A- CAC PHÖÔNG PHAP CHÍNH XAC

1- PHÖÔNG PHAP KHÖ GAUSS (PHAN TÖ TRU)

2- PHAN TÍCH NHAN TÖ A = LU

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B- CAC PHÖÔNG PHAP LAP

1- LAP JACOBI

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TONG QUAN

Hei n phöông trình bac 1 (tuyen tính), n an $\rightarrow$ Dang Ax = b:

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}, \quad b = \begin{bmatrix} b_1 \\ b_2 \\ \ldots \\ b_n \end{bmatrix}, \quad x = \begin{bmatrix} x_1 \\ x_2 \\ \ldots \\ x_n \end{bmatrix} = [x_1 \cdots x_n]^T$$

Ñôn gian: Hei tam giac $\quad A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & a_{22} & \cdots & a_{2n} \\ \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots \\ 0 & 0 & \cdots & a_{nn} \end{bmatrix}$ $\Rightarrow$ Giai lui

Hang i: $h_i = [a_{i1}\ a_{i2}\ \ldots\ a_{in}]^T$. Bien ñoi sô cap tren hang $h_i \rightarrow h_i + kh_j$: Nhan h_j vôi k roi cong xuong h_i (chæ h_i thay ñoi)

PHÖÔNG PHAÙP KHÖÛ GAUSS

Giaûi thuaät: Bieán ñoåi sô caáp treân haøng → A: Δ treân → Giaûi lui

VD: Giaûi heä
$$\begin{cases} 2x_1 - 2x_2 + 3x_3 = 1 \\ 6x_1 - 7x_2 + 14x_3 = 5 \\ 4x_1 - 8x_2 + 30x_3 = 14 \end{cases}$$

Xaây döïng ma traän môû roäng

$$\overline{A} = [A \mid b] = \left[\begin{array}{ccc|c} 2 & -2 & 3 & 1 \\ 6 & -7 & 14 & 5 \\ 4 & -8 & 30 & 14 \end{array}\right]$$

Khöû coät 1 vôùi heä soá khöû m_{1j}

$$m_{1j} = \frac{a_{1j}}{a_{11}} \quad \text{Toång quaùt : } m_{ij} = \frac{a_{ij}}{a_{ii}}$$

$$m_{21} = \frac{6}{2} = 3$$

$$m_{31} = \frac{4}{2} = 2$$

$$\Rightarrow \left[\begin{array}{ccc|c} 2 & -2 & 3 & 1 \\ 6 & -7 & 14 & 5 \\ 4 & -8 & 30 & 14 \end{array}\right] \begin{array}{c} h_2 \to h_2 - 3h_1 \\ \to \\ h_3 \to h_3 - 2h_1 \end{array} \left[\begin{array}{ccc|c} 2 & -2 & 3 & 1 \\ 0 & -1 & 5 & 2 \\ 0 & -4 & 24 & 12 \end{array}\right]$$

GIAI LUI & PHAN TÖ TRU

Khö cot 2 vôi he so khö: $m_{32} = \dfrac{-4}{-1} = 4$

$$\left[\begin{array}{ccc|c} 2 & -2 & 3 & 1 \\ 0 & -1 & 5 & 2 \\ 0 & -4 & 24 & 12 \end{array}\right] \xrightarrow{h_3 \to h_3 - 4h_2} \left[\begin{array}{ccc|c} 2 & -2 & 3 & 1 \\ 0 & -1 & 5 & 2 \\ 0 & 0 & 4 & 4 \end{array}\right]$$

Giai lui vôi he tam giac tren thu ñöôc:

$$\begin{cases} 2x_1 - 2x_2 + 3x_3 = 1 \\ \quad\; - x_2 + 5x_3 = 2 \\ \qquad\quad 4x_3 = 4 \end{cases} \Rightarrow \begin{cases} x_3 = 4/4 = 1 \\ x_2 = (2 - 5x_3)/(-1) = 3 \\ x_1 = (1 + 2x_2 - 3x_3)/2 = 2 \end{cases} \Rightarrow x = \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}$$

Ñieu kien: Khö cot 1: $a_{11}^{(1)} \neq 0$ & Khö cot 2: $a_{22}^{(2)} \neq 0$ &

Giai lui: $a_{33}^{(3)} \neq 0 \Rightarrow$ $\boxed{\text{Phan tö tru (pivot) } a_{kk} \neq 0}$

KHÖ GAUSS VÔI LENH MAPLE

> with(linalg);   # Khôi ñoing goi leinh Ñai soi tuyein tính

> A := matrix(2,3,[2, 3, 4, 1, 2, 3]); # Nhaip ma train

> m21 := A[2,1]/A[1,1]; # Tính hei soi khöi

> A := addrow(A,1,2,–m21) ; # Coing haing $h_2 \rightarrow h_2 - m_{21}h_1$

> A := swaprow(A,1,2) ; # Neiu cain thieit, ñoi haing $h_2 \leftrightarrow h_1$

> x := backsup(A) ; # Hei ña ôi daing tam giac trein: Giai lui

> AA := gausselim(A); # Leinh goip khöi Gauss toan ma train

VD: Giai hei

$$
\begin{cases}
x_1 - x_2 + 2x_3 - x_4 = -8 \\
2x_1 - 2x_2 + 3x_3 - 3x_4 = -20 \\
x_1 + x_2 + x_3 = -2 \\
x_1 - x_2 + 4x_3 + 3x_4 = 4
\end{cases}
$$

KHÖ GAUSS VÔI MA TRAN “LE”: PIVOT ÑÔN VÒ

VD: Giai he vôi phep khö Gauss, lam tron 3 chö so le)

$$\begin{cases} 2.08x_1 - 1.3x_2 = 0.608 \\ -0.7x_1 + 2.08x_2 - 1.3x_3 = -0.152 \\ -0.7x_2 + 2.08x_3 - 1.3x_3 = -0.168 \\ -0.7x_3 + 2.08x_4 = 1.116 \end{cases}$$

$$\Rightarrow [A|b] = \left[\begin{array}{cccc|c} 2.08 & -1.3 & 0 & 0 & 1.264 \\ -0.7 & 2.08 & -1.3 & 0 & -0.152 \\ 0 & -0.7 & 2.08 & -1.3 & -0.168 \\ 0 & 0 & -0.7 & 2.08 & 1.116 \end{array}\right]$$

$$\Rightarrow y = \begin{bmatrix} 1.006 \\ 0.636 \\ 0.593 \\ 0.736 \end{bmatrix}$$

THÖC TE TÍNH TOAN: VAN ÑE LAM TRON SO

VD: Giai he tren may tính vôi phep lam tron 4 chö co nghóa

$$\begin{cases} 0.003x_1 + 59.14x_2 = 59.17 \ (\mathrm{E}_1) \\ 5.291x_1 - 6.130x_2 = 46.78 \ (\mathrm{E}_2) \end{cases}$$

Nghiem chính xac: $[10, 1]^T$

Quy tac lam tron tren may tính: Lam tron chö so co nghóa

$$12{,}34567 = 1{,}234567 \cdot 10^1 \approx 1{,}235 \cdot 10^1 = 12{,}35$$

Tru khö: $a_{11} = 0.003 \neq 0 \Rightarrow$ $\quad m_{21} = \dfrac{a_{21}}{a_{11}} = 1763{,}67 \approx 1764$

Bien ñoi cot mot: $(E_2) \rightarrow (E_2) - m_{21}(E_1)$

$$\begin{cases} 0.003x_1 + 59.14x_2 = 59.17 \\ \qquad\qquad -104300x_2 = -104400 \end{cases} \Rightarrow \begin{cases} x_2 = 1.001 \\ x_1 = -10 \end{cases} : \text{Tai sao ???}$$

PHAN TÍCH NHAN TÖ (MATRIX FACTORIZATIONS)

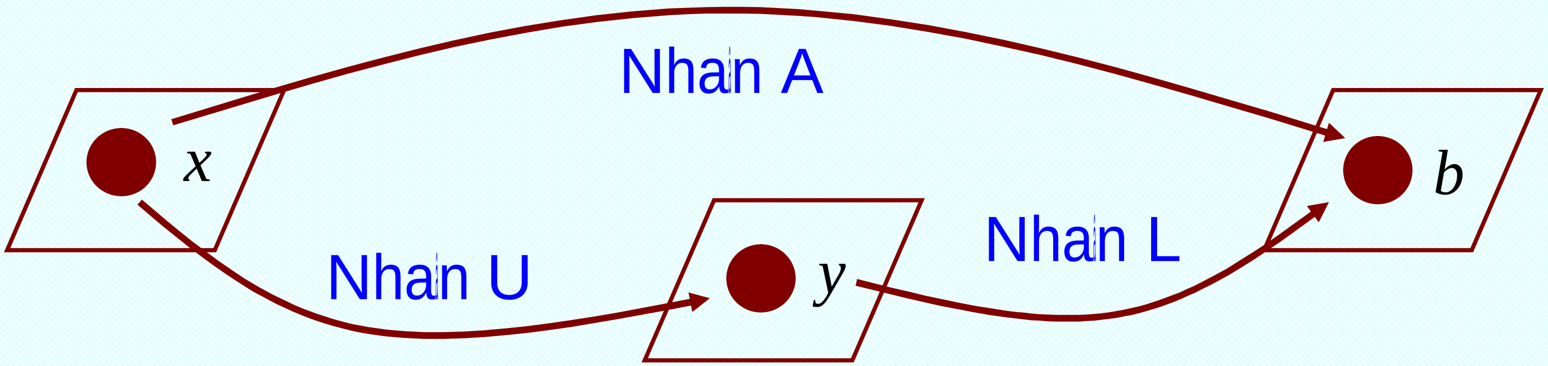
Ma tran vuong A phan tích ñöôc thanh dang LU ⇔

$$A = \underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 \\ * & 1 & 0 & 0 \\ * & * & 1 & 0 \\ * & * & * & 1 \end{bmatrix}}_{L} \cdot \underbrace{\begin{bmatrix} * & * & * & * \\ 0 & * & * & * \\ 0 & 0 & * & * \\ 0 & 0 & 0 & * \end{bmatrix}}_{U}$$

He Ax = b ⇔ (LU)x = b ⇔

$$\begin{cases} Ux = y \ \ (1) \\ Ly = b \ \ (2) \end{cases}$$: 2 he tam giac

Giai he ñau ⇔ Giai 2 he Δ: Ly = b (2) tìm y; Ux = y (1) tìm x



VÍ DU

Giai söi ma tran A phan tích ñöôc thanh dang LU nhö sau:

$$A = \begin{bmatrix} 3 & -7 & -2 & 2 \\ -3 & 5 & 1 & 0 \\ 6 & -4 & 0 & -5 \\ -9 & 5 & -5 & 12 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 2 & -5 & 1 & 0 \\ -3 & 8 & 3 & 1 \end{bmatrix} . \begin{bmatrix} 3 & -7 & -2 & 1 \\ 0 & -2 & -1 & 1 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

Söi dung phan tích LU tren giai he Ax = b = $[-9\ \ 5\ \ 7\ \ 11]^T$

Giai Ly = b tìm y

Giai Ux = y tìm x

PHAN TÍCH NHAN TÖ A = LU

Quan sat: Ma tran khö L va ma tran ket qua U. Xet tích L.U

$$\begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 2 & 4 & 1 \end{bmatrix} \cdot \begin{bmatrix} 2 & -2 & 3 \\ 0 & -1 & 5 \\ 0 & 0 & 4 \end{bmatrix} =$$

Ket qua: Neu qua trình khö Gauss dien ra bình thöông (khong ñoi hang), ma tran A cua he Ax = b phan tích ñöôc thanh tích LU: A = LU vôi

- ➢ L (lower): ma tran tam giac döôi, ñöông cheo chính bang 1, chöa cac he so khö ô vò trí khö
- ➢ U (upper): ma tran tam giac tren, cung la ma tran ket qua nhan ñöôc sau qua trình khö Gauss

GIAI THUAT TÌM LU (CROUT – DOOLITLE)

Phan tích LU vôi ñöông cheo chính L bang 1 $\Rightarrow$ Khöi Gauss (khong ñoi hang). Cac he so khöi tao L, ma tran ket qua: U

VD: $A = \begin{bmatrix} 2 & -2 & 3 \\ 6 & -7 & 14 \\ 4 & -8 & 30 \end{bmatrix} : \begin{matrix} m_{21} = 3 \\ m_{31} = 2 \\ m_{32} = 4 \end{matrix} \Rightarrow L = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 2 & 4 & 1 \end{bmatrix}$

L (hoac U) co ñcheo chính = 1 $\Rightarrow$ G/thuat Doolitle (Crout)

For j = 1 to n: i = 1 $\rightarrow$ j $\quad u_{ij} = a_{ij} - \sum_{k=1}^{i-1} l_{ik} u_{kj}$

i = j +1 $\rightarrow$ n $\quad l_{ij} = \frac{1}{u_{jj}} \left[a_{ij} - \sum_{k=1}^{j-1} l_{ik} u_{kj} \right]$

Töi xem SGK/ 35

MINH HOA GIAI THUAT DOOLITLE (ÑCHEO L = 1)

VD: $A = \begin{bmatrix} 2 & -2 & 3 \\ 6 & -7 & 14 \\ 4 & -8 & 30 \end{bmatrix} \qquad \Rightarrow L = \begin{bmatrix} 1 & 0 & 0 \\ & 1 & 0 \\ & & 1 \end{bmatrix}, U = \begin{bmatrix} & & \\ 0 & & \\ 0 & 0 & \end{bmatrix}$

j = 1: i = 1 $u_{11} = a_{11}$

i = 2 $\ell_{21} = \dfrac{a_{21}}{u_{11}}$ i = 3 $\ell_{31} = \dfrac{a_{31}}{u_{11}}$

j = 2: i = 1 $u_{12} = a_{12}$ i = 2 $u_{22} = a_{22} - \ell_{21} u_{12}$

i = 3 $\ell_{32} = \dfrac{a_{32} - \ell_{31} u_{12}}{u_{22}}$

j = 3: i = 1 $u_{13} = a_{13}$ i = 2 $u_{23} = a_{23} - \ell_{21} u_{13}$ i = 3

i = 3 $u_{33} = a_{33} - \ell_{31} u_{13} - \ell_{32} u_{23}$

PHAN TÍCH CHOLESKY

Tööng tö phan tích LU nhöng gon hôn “phan nöa”!

Ma tran vuong A (n hang, n cot) : A = [a_{ij}] xac ñònh döông

$$\forall\ x = [x_1, \ldots, x_n]^T \in R^n, x \neq 0 : x^T A x = \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij} x_i x_j > 0$$

VD: $A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 5 & -2 \\ 0 & -2 & 2 \end{bmatrix}$: ma tran (ñoi xöng) xac ñònh döông vì

$$\forall\ x = [x_1, x_2, x_3]^T \in R^3 \Rightarrow x^T A x = x_1^2 + 5x_2^2 + 2x_3^2 + 2x_1x_2 - 4x_2x_3 > 0$$

A: XÑD $\Leftrightarrow$ n ñònh thöc con (hvuong) tren ñ/cheo chính ñeu > 0

GIAI THUAT CHOLESKY

Ñònh ly: Ma tran A ñoi xöng xac ñònh döông ⇔ Ton tai ma tran tam giac döôi B thoa man : $A = BB^T$

A ñoi xöng: For i = 1 to n do

$$b_{ii} = \sqrt{a_{ii} - \sum_{k=1}^{i-1} b_{ik}^2}$$

For j = i+1 to n do

$$b_{ji} = \frac{1}{b_{ii}}\left[a_{ji} - \sum_{k=1}^{i-1} b_{jk} b_{ik}\right]$$

$Ax = b \Leftrightarrow (BB^T)x = b \Leftrightarrow B^Tx = y$ & $By = b$: 2 he Δ (nhö LU)

A k0 xac ñònh döông (chæ ñoi xöng): $A = BB^T$ co the chöa so phöc

$\Rightarrow$ 2 he $B^Tx = y$ & $By = b$: phöc. Nhöng nghiem x: thöc!

MINH HOẠ GIẢI THUẬT CHOLESKY

VD: $A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 5 & -2 \\ 0 & -2 & 2 \end{bmatrix} \quad \Rightarrow B = \begin{bmatrix} & 0 & 0 \\ & & 0 \\ & & \end{bmatrix}$

i = 1 $\quad b_{11} = \sqrt{a_{11}}$

j = 2 $\quad b_{21} = \dfrac{a_{21}}{b_{11}}$ $\qquad$ j = 3 $\quad b_{31} = \dfrac{a_{31}}{b_{11}}$

i = 2 $\quad b_{22} = \sqrt{a_{22} - b_{21}^2}$

j = 3 $\quad b_{32} = \dfrac{a_{32} - b_{31}b_{21}}{b_{22}}$

i = 3 $\quad b_{33} = \sqrt{a_{33} - b_{31}^2 - b_{32}^2}$

TONG QUAN PHÖÔNG PHAP LAP

Chöông 1: Phöông phap lap ñôn vôi phöông trình f(x) = 0

$$f(x)=0 \;\Leftrightarrow\; \begin{cases} x=\varphi(x) \\ |\varphi(x)-\varphi(y)|\le q|x-y| : |\varphi'(x)|\le q<1 \end{cases} \;\Rightarrow\; x_{n+1}=\varphi(x_n)$$

Hei $Ax = b \Leftrightarrow x = Tx + c = \varphi(x)$, T: ma tran, c: vectô. Ñkien:

$\|\varphi(x) - \varphi(y)\| \le q\|x - y\| \Rightarrow$ Day lap: $x^{(n+1)} = Tx^{(n)} + c$

Chuan vectô, ma tran: $x = [x_1, x_2 \ldots x_n]^T \in R^n$, $A = [a_{ij}]$

$$\|x\|_\infty = \max_{1\le i\le n}\{|x_i|\} \Rightarrow \|A\|_\infty = \max_{1\le i\le n}\left\{\sum_{j=1}^{n}|a_{ij}|\right\}$$

$$\|x\|_1 = \sum_{i=1}^{n}|x_i| \Rightarrow \|A\|_1 = \max_{1\le j\le n}\left\{\sum_{i=1}^{n}|a_{ij}|\right\}$$

VÍ DUÏ

- Tính caùc chuaån vectô vaø ma traän

$$x=\begin{bmatrix}1\\-3\\-2\end{bmatrix}\Rightarrow\begin{cases}\|x\|_\infty = 3\\ \|x\|_1 = 6\end{cases}\qquad A=\begin{bmatrix}-1&3&2\\-2&4&3\\-5&6&-1\end{bmatrix}\Rightarrow\begin{cases}\|A\|_\infty = 12\\ \|A\|_1 = 13\end{cases}$$

- Vectô naøo trong soá hai vectô sau xaáp xæ toát nhaát theo chuaån ∞, chuaån moät nghieäm heä phöông trình

$$x^{(1)}=\begin{bmatrix}-0.1\\-7.4\\5.2\end{bmatrix},\ x^{(2)}=\begin{bmatrix}0.3\\-6.8\\4.7\end{bmatrix}\qquad\begin{cases}x_1+2x_2+3x_3=1\\2x_1+3x_2+4x_3=-1\\3x_1+4x_2+6x_3=2\end{cases}\ \rightarrow\begin{cases}x^{(2)}\overset{\|\cdot\|_\infty}{\approx}x\\ x^{(1)}\overset{\|\cdot\|_1}{\approx}x\end{cases}$$

Tch. chuaån vectô, chuaån ma traän: Chuaån tích ≤ tích chuaån

$$\|Ax\|\le\|A\|\cdot\|x\|\quad\Rightarrow\quad\|\varphi(x)-\varphi(y)\|=\|T(x-y)\|\le\|T\|\cdot\|x-y\|:\|T\|<1$$

LAP JACOBI

Vôi vectô $x^{(0)} = [0, 0, 0]^T$, tìm vectô nghiem xap xæ $x^{(k)}$ cua phep lap Jacobi vôi he sau. Döng: $x^{(k)}$ "giong" $x^{(k-1)}$ (khoang 0.3)

$$\begin{cases} 10x_1 + 3x_2 + x_3 = 7.5 \\ -3x_1 + 10x_2 - x_3 = 9 \\ -x_1 + 2x_2 + 8x_3 = -2.5 \end{cases}$$

. So vôi nghiem $\alpha = [0.5, 1, -0.5]^T$

1/ Rut x tren ñöông cheo chính $\Rightarrow$ Ñöa ve dang x = Tx + c

$$\begin{cases} x_1 = -\frac{3}{10}x_2 - \frac{1}{10}x_3 + \frac{7.5}{10} = -0.3x_2 - 0.1x_3 + 0.75 \\ x_2 = \frac{3}{10}x_1 + \frac{1}{10}x_3 + \frac{9}{10} = 0.3x_1 + 0.1x_3 + 0.9 \\ x_3 = \frac{1}{8}x_1 - \frac{2}{8}x_2 - \frac{2.5}{8} = 0.125x_1 - 0.25x_2 - 0.3125 \end{cases} \Leftrightarrow \boxed{x = Tx + c}$$

CONG THÖC LAP JACOBI

$$T=\begin{bmatrix} 0 & -\frac{3}{10} & -\frac{1}{10} \\ \frac{3}{10} & 0 & \frac{1}{10} \\ \frac{1}{8} & -\frac{2}{8} & 0 \end{bmatrix},\quad \|T\|_\infty=\max\left\{\frac{4}{10},\frac{3}{8}\right\}=\frac{4}{10}<1,\quad c=\begin{bmatrix} 0.75 \\ 0.9 \\ -0.3125 \end{bmatrix}$$

2/ Tö $x^{(0)}$ tính $x^{(1)}$:

$$\begin{cases} x_1^{(1)}=-0.3x_2^{(0)}-0.1x_3^{(0)}+0.75= \ 0.75 \\ x_2^{(1)}=0.3x_1^{(0)}+0.1x_3^{(0)}+0.9= \ 0.9 \\ x_3^{(1)}=0.125x_1^{(0)}-0.25x_2^{(0)}-0.3125= \ -0.3125 \end{cases}$$

Tong quat: $x^{(k)} \Rightarrow x^{(k+1)}$:

$$\begin{cases} x_1^{(k+1)}=-0.3x_2^{(k)}-0.1x_3^{(k)}+0.75 \\ x_2^{(k+1)}=0.3x_1^{(k)}+0.1x_3^{(k)}+0.9 \\ x_3^{(k+1)}=0.125x_1^{(k)}-0.25x_2^{(k)}-0.3125 \end{cases}$$

Sai so: Nhö lap ñôn vôi q = $\|T\|_\infty$: $\left\|x^{(k)}-x^*\right\|_\infty \le \frac{q}{1-q}\left\|x^{(1)}-x^{(0)}\right\|_\infty$

LAP JACOBI KHONG BIEN ÑOI MA TRAN A

He Ax = b:
$$\begin{cases}10x_1+3x_2+x_3=7.5\\-3x_1+10x_2-x_3=9\\-x_1+2x_2+8x_3=-2.5\end{cases}\ \overset{\text{Ñ/cheo}}{\underset{\text{chính}}{\Leftrightarrow}}\ \begin{cases}10x_1= & -3x_2-x_3+7.5\\10x_2=3x_1 & +x_3+9\\8x_3=x_1-2x_2 & -2.5\end{cases}$$

$$\begin{cases}10x_1^{(k+1)}= & -3x_2^{(k)}-x_3^{(k)}+7.5\\10x_2^{(k+1)}=3x_1^{(k)} & +x_3^{(k)}+9\\8x_3^{(k+1)}=x_1^{(k)}-2x_2^{(k)} & -2.5\end{cases}$$

: Giö ñ/cheo chính ô ve trai ($\rightarrow x^{(k+1)}$) ; Chuyen so hang con lai sang ve phai ($\rightarrow x^{(k)}$)

$$\begin{cases}10x_1^{(k+1)}+3x_2^{(k)}+x_3^{(k)}=7.5\\-3x_1^{(k)}+10x_2^{(k+1)}-x_3^{(k)}=9\\-x_1^{(k)}+2x_2^{(k)}+8x_3^{(k+1)}=-2.5\end{cases}$$

: Thay $x^{(k)}$ vao cac so hang ngoai ñöông cheo chính. Xem $x^{(k+1)}$ la an. Giai $\rightarrow x^{(k+1)}$

TÍNH TOAN & KET QUA LAP JACOBI

Hei: $\begin{cases} 10x_1 + 3x_2 + x_3 = 7.5 \\ -3x_1 + 10x_2 - x_3 = 9 \\ -x_1 + 2x_2 + 8x_3 = -2.5 \end{cases}$ $\Leftrightarrow$ Lap Jacobi : $\begin{cases} 10x_1^{(k+1)} = -3x_2^{(k)} - x_3^{(k)} + 7.5 \\ 10x_2^{(k+1)} = 3x_1^{(k)} + x_3^{(k)} + 9 \\ 8x_3^{(k+1)} = x_1^{(k)} - 2x_2^{(k)} - 2.5 \end{cases}$

$$\begin{cases} x_1^{(k+1)} = \dfrac{-3x_2^{(k)} - x_3^{(k)} + 7.5}{10} \\ x_2^{(k+1)} = \dfrac{3x_1^{(k)} + x_3^{(k)} + 9}{10} \\ x_3^{(k+1)} = \dfrac{x_1^{(k)} - 2x_2^{(k)} - 2.5}{8} \end{cases}$$

k	0	1	2
$x_1^{(k)}$	0.0	0.75	
$x_2^{(k)}$	0.0	0.9	
$x_3^{(k)}$	0.0	–0.3125	
$\|x^{(k)}-x^{(k-1)}\|_\infty$		0.9	

Öu ñiem Lap Jacobi: Giai cac he “thöa” (chöa rat nhieu so 0)

ÑK ñu : $\|T\|_\infty < 1 \Leftrightarrow$ M/tran ñ/c troi nghiem ngat: $|a_{ii}| > \sum_{j=1,\, j\neq i}^{n} |a_{ij}| \ \forall\, i$

LAP GAUSS – SEIDEL

Töông töi lap Jacobi nhöng vôi thong tin cap nhat hoai

$$\text{He}: \begin{cases} 10x_1 + 3x_2 + x_3 = 7.5 \\ -3x_1 + 10x_2 - x_3 = 9 \\ -x_1 + 2x_2 + 8x_3 = -2.5 \end{cases} \overset{\text{Lap}}{\underset{\text{Jacobi}}{\Leftrightarrow}} \begin{cases} x_1^{(k+1)} = \dfrac{-3x_2^{(k)} - x_3^{(k)} + 7.5}{10} \\ x_2^{(k+1)} = \dfrac{3x_1^{(k)} + x_3^{(k)} + 9}{10} \\ x_3^{(k+1)} = \dfrac{x_1^{(k)} - 2x_2^{(k)} - 2.5}{8} \end{cases}$$

Dung $x^{(k)}$ ñe tính gia trò cua $x^{(k+1)}$

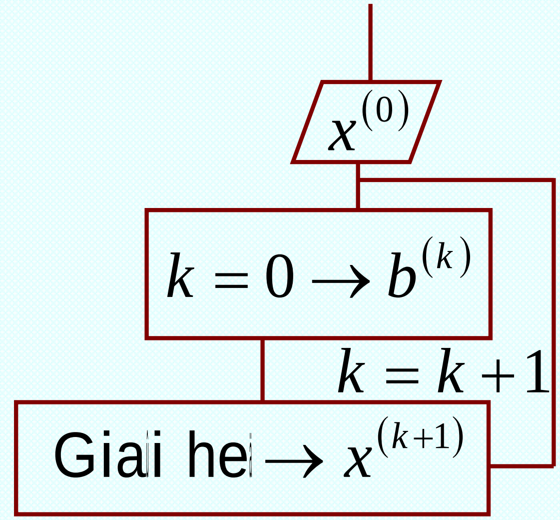
$$\overset{\text{Gauss}}{\underset{\text{Seidel}}{\Rightarrow}} \begin{cases} x_1^{(k+1)} = \dfrac{-3x_2^{(k)} - x_3^{(k)} + 7.5}{10} \\ x_2^{(k+1)} = \dfrac{3x_1^{(k+1)} + x_3^{(k)} + 9}{10} \\ x_3^{(k+1)} = \dfrac{x_1^{(k+1)} - 2x_2^{(k+1)} - 2.5}{8} \end{cases}$$

- ❖ x_1 (môi): dung x_2 (cu), x_3 (cu)
- ❖ x_2 (môi): dung x_1 (môi), x_3 (cu)
- ❖ x_3 (môi): dung x_1 (môi), x_2 (môi)

LAP GAUSS – SEIDEL: SÔ ÑO TACH MA TRAN

Trình bay dang khac: Xem $x^{(k+1)}$ la an va chuyen sang ve trai

$$\begin{cases} 10x_1 + 3x_2 + x_3 = 7.5 \\ -3x_1 + 10x_2 - x_3 = 9 \\ -x_1 + 2x_2 + 8x_3 = -2.5 \end{cases} \Leftrightarrow \begin{cases} 10x_1^{(k+1)} = -3x_2^{(k)} - x_3^{(k)} + 7.5 \\ -3x_1^{(k+1)} + 10x_2^{(k+1)} = x_3^{(k)} + 9 \\ -x_1^{(k+1)} + 2x_2^{(k+1)} + 8x_3^{(k+1)} = -2.5 \end{cases}$$



Gauss - Seidel: Biet $x^{(k)}$ $\rightarrow$ Tính ve phai $b^{(k)}$ $\rightarrow$ Giai he ra $x^{(k+1)}$

LAIP GAUSS – SEIDEL: VÍ DUI TAICH MA TRAIN

Xeit ví dui laip Gauss – Seidel, $x^{(0)} = [0, 0, 0]^T$. Coing thöic laip:

$$\left.\begin{cases} 10x_1^{(k+1)} & = -3x_2^{(k)} - x_3^{(k)} + 7.5 \\ -3x_1^{(k+1)} + 10x_2^{(k+1)} & = x_3^{(k)} + 9 \\ -x_1^{(k+1)} + 2x_2^{(k+1)} + 8x_3^{(k+1)} & = -2.5 \end{cases}\right\} \rightarrow b^{(k)}$$

k	0		1		2	
	x	b	x	b	x	b
$x_1^{(k)}$	0.0					
$x_2^{(k)}$	0.0					
$x_3^{(k)}$	0.0					
$\|\|x^{(k)}-x^{(k-1)}\|\|_\infty$						

Pheip laip ⇔ Thay hei Ax = b baing giaii liein tieip nhieiu hei Δ

TONG KET LAP JACOBI & GAUSS – SEIDEL

$$\begin{cases} 10x_1 + 3x_2 + x_3 = 7.5 \\ -3x_1 + 10x_2 - x_3 = 9 \\ -x_1 + 2x_2 + 8x_3 = -2.5 \end{cases}$$

x = Tx + c

$$\begin{cases} 10x_1 = \quad -3x_2 - x_3 + 7.5 \\ 10x_2 = 3x_1 \quad + x_3 + 9 \\ 8x_3 = x_1 - 2x_2 \quad - 2.5 \end{cases}$$

Lap Jacobi

$$\begin{cases} 10x_1^{(k+1)} = \quad -3x_2^{(k)} - x_3^{(k)} + 7.5 \\ 10x_2^{(k+1)} = 3x_1^{(k)} \quad + x_3^{(k)} + 9 \\ 8x_3^{(k+1)} = x_1^{(k)} - 2x_2^{(k)} \quad - 2.5 \end{cases}$$

Lap Gauss– Seidel

$$\begin{cases} 10x_1^{(k+1)} = -3x_2^{(k)} - x_3^{(k)} + 7.5 \\ 10x_2^{(k+1)} - 3x_1^{(k+1)} = x_3^{(k)} + 9 \\ 8x_3^{(k+1)} - x_1^{(k+1)} + 2x_2^{(k+1)} = -2.5 \end{cases}$$

HE PHÖÔNG TRÌNH BÒ NHIEU

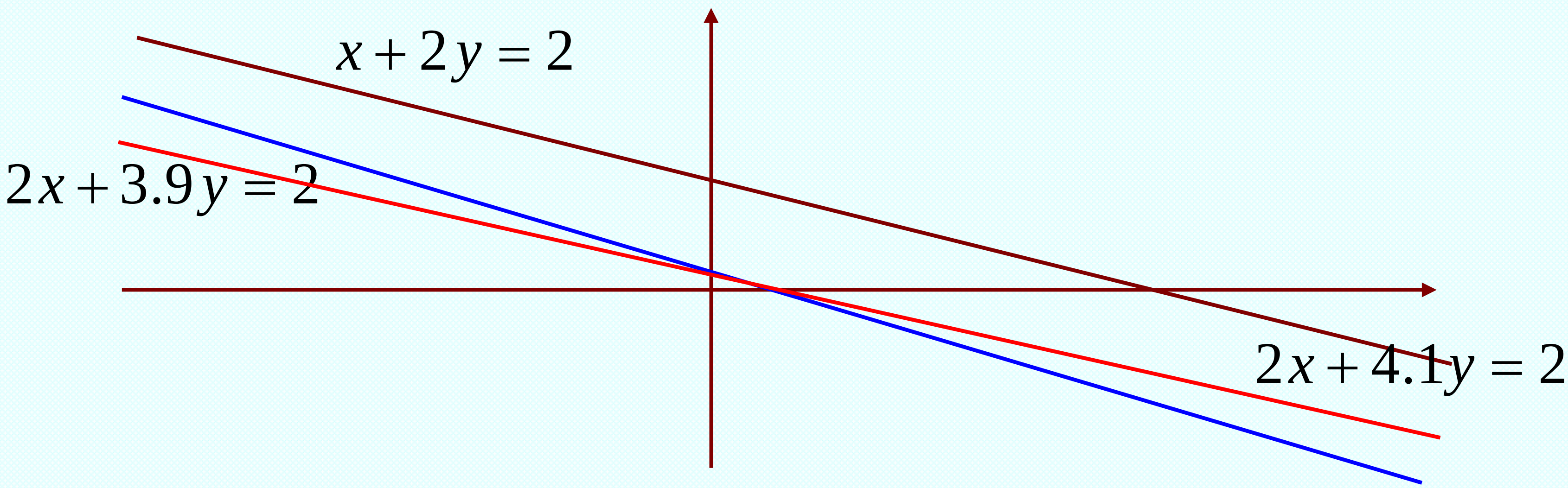
Minh hoa: Gia i 2 he phöông trình va nha n xet

$$\begin{cases} x + 2y = 2 \\ 2x + 3.9y = 2 \end{cases}$$

He "gan" nhau, nghiem "xa" nhau! Do detA $\approx$ 0:

$$\begin{cases} x + 2y = 2 \\ 2x + 4.1y = 2 \end{cases}$$

$$x_i = \frac{\det A_i}{\det A}?$$



VÍ DỤ WILSON: Ax = b, detA = 1

$$Ax = b: \ A = \begin{bmatrix} 10 & 7 & 8 & 7 \\ 7 & 5 & 6 & 5 \\ 8 & 6 & 10 & 9 \\ 7 & 5 & 9 & 10 \end{bmatrix}, b = \begin{bmatrix} 32 \\ 23 \\ 33 \\ 31 \end{bmatrix} \Rightarrow x = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

$$A(x + \delta x) = b + \delta b: \ b + \delta b = \begin{bmatrix} 32.1 \\ 22.9 \\ 33.1 \\ 30.9 \end{bmatrix} \Rightarrow x + \delta x =$$

$$(A + \Delta A)\left(\underbrace{x + \Delta x}_{x'}\right) = b: A + \Delta A = \begin{bmatrix} 10 & 7 & 8.1 & 7.2 \\ 7.08 & 5.04 & 6 & 5 \\ 8 & 5.98 & 9.89 & 9 \\ 6.99 & 4.99 & 9 & 9.98 \end{bmatrix} \Rightarrow$$

SO ÑIEU KIEN CUA HE Ax = b

- “Nhieu” ve phai A(x + δx) = b + δb ⇒ $\dfrac{\|\delta x\|}{\|x\|} >> \dfrac{\|\delta b\|}{\|b\|}$

- “Nhieu” ve trai (A + ΔA)(x + Δx) = b ⇒ $\dfrac{\|\Delta x\|}{\|x+\Delta x\|} >> \dfrac{\|\Delta A\|}{\|A+\Delta A\|}$

$$\frac{\|\delta x\|}{\|x\|} \le \left\{\|A\|.\|A^{-1}\|\right\} \frac{\|\delta b\|}{\|b\|}; \qquad \frac{\|\Delta x\|}{\|x+\Delta x\|} \le \left\{\|A\|.\|A^{-1}\|\right\} \frac{\|\Delta A\|}{\|A+\Delta A\|}$$

So ñieu kien: $\kappa_\infty(A) = \|A\|_\infty \,.\, \|A^{-1}\|_\infty$ ñac tröng cho ño “nhay cam” cua nghiem he Ax = b ñoi vôi nhöng thay ñoi du rat nho tren b hoac A

He ñieu kien xau (ill – conditionned): $\kappa_\infty(A) >> 1$

VÍ DU

VD Wilson:

$$A=\begin{bmatrix}10 & 7 & 8 & 7\\ 7 & 5 & 6 & 5\\ 8 & 6 & 10 & 9\\ 7 & 5 & 9 & 10\end{bmatrix}$$

$$\Rightarrow A^{-1}=\begin{bmatrix}25 & -41 & 10 & -6\\ -41 & 68 & -17 & 10\\ 10 & -17 & 5 & -3\\ -6 & 10 & -3 & 2\end{bmatrix}$$

VD: Tính so ñieu kien theo chuan vo cung $\kappa_\infty(A)$ cua ma tran

$$A=\begin{bmatrix}1.003 & 58.09\\ 5.550 & 321.8\end{bmatrix}$$

PHÖÔNG PHAP TÌM MA TRAN NGÖÔC

Tính ma tran ngöôc $A^{-1} = \begin{bmatrix} x & y \\ z & t \end{bmatrix} = \begin{bmatrix} c_1 & c_2 \end{bmatrix}$

- Giai he phöông trình $A.c_1 = e_1 = [1\ 0]^T$ (vectô ñôn vò thöù nhat) tren may tính bo tui $\Rightarrow$ Cot 1 cua ma tran ngöôc
- Van trong che ño giai he phöông trình, giai tiep he $A.c_2 = e_2 = [0\ 1]^T$ (vectô ñôn vò thöù nhì) $\Rightarrow$ Cot 2 cua A^{-1}
- Tröông hôp ma tran cap 3: Giai 3 he $Ac_1 = e_1$, $Ac_2 = e_2$, $Ac_3 = e_3$ vôi e_1, e_2, e_3 lan löôt la 3 vectô ñôn vò $\Rightarrow$ Tìm ñöôc 3 vectô nghiem c_1, c_2, c_3: 3 cot cua ma tran ngöôc A^{-1} can tìm